$\mathcal{N}=1$ SQM

superspace $t, \theta, \bar\theta$ odd

$$Q = \frac{\partial}{\partial\theta} + \bar\theta\frac{\partial}{\partial t}, \quad \bar Q = \frac{\partial}{\partial\bar\theta} + \theta\frac{\partial}{\partial t}$$

$$D = \frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial t}, \quad \bar D = \frac{\partial}{\partial\bar\theta} - \bar\theta\frac{\partial}{\partial t}$$

$$\Phi^i = X^i(t) + \theta\psi^i + \bar\theta\bar\psi^i + \theta\bar\theta F^i \quad (1)$$

$$Q\Phi^i = \psi^i + \bar\theta\partial_t X^i + \bar\theta F^i + \bar\theta\theta\partial_t\psi^i \quad (2)$$

$\delta_Q \psi = 0$

$\delta_Q X^i = \psi^i \quad \delta_Q\bar\psi^i = \partial_t X^i + F^i \quad \delta_Q F^i = \partial_t\psi^i$

$\delta_{\bar Q} X^i = \bar\psi^i \quad \delta_{\bar Q}\psi^i = \partial_t X^i - F^i \quad \delta_{\bar Q} F^i = \partial_t\bar\psi^i$

$$D\Phi^i = \psi^i - \bar\theta\partial_t X^i + \bar\theta F^i - \bar\theta\theta\partial_t\psi^i$$
$$\bar D\Phi^i = \bar\psi^i - \theta\partial_t X^i - \theta F^i - \bar\theta\theta\partial_t\bar\psi^i \quad (3)$$

$$\int d^2\theta\, g_{ij}(\Phi)\, D\Phi^i \bar D\Phi^j$$

<u>Case 1.</u> g_{ij} is constant

pick up 2 θ's $\quad g_{ij}(\partial_t X^i + F^i)(\partial_t X^j - F^j) =$

$$= \underline{g_{ij}\partial_t X^i\partial_t X^j - g_{ij}F^iF^j}$$

From terms in (3) linear in θ.

I can pick term $\theta\bar\theta \to$

$$\underline{g_{ij}\bar\psi^i\partial_t\psi^j}$$

Kinetic terms

I can add so-called superpotential.

$$\int d^2\theta\, W(\Phi) = \int \frac{\partial W}{\partial X^i}F^i\bar\theta\theta +$$

$$+ \int \frac{\partial^2 W}{\partial X^i\partial X^j}\bar\psi^i\psi^j\bar\theta\theta$$

Alltogether I get

$$S = \int \left(g_{ij} \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} + g_{ij} \bar{\psi}^i \frac{\partial}{\partial t} \psi^j + \boxed{g_{ij} F^i F^j} + \boxed{\frac{\partial W}{\partial x^i} F^i} + \frac{\partial^2 W}{\partial x^i \partial x^j} \bar{\psi}^i \psi^j \right) dt$$

Field F has no time derivative in the action, so it could be eliminated by gaussian integration.

Result is $\boxed{g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}}$

We have a theory that is:

Free theory for bosons $x^i(t)$ with potential $g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}$ and.

Theory of fermions $g_{ij} \bar{\psi}^i \frac{\partial}{\partial t} \psi^j$ interacting with bosons by

$$\frac{\partial^2 W}{\partial x^i \partial x^j} \bar{\psi}^i \psi^j$$

Let me go to interpretation of such theory in functorial formalism.

1) About bosons $\rightarrow \int\limits_{\substack{x(0)=x_0 \\ x(T)=x_1}} \mathcal{D}x \exp \int g_{ij} \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} - U(x)$

Fun $(X) \times$ Fun (X)

2) Fermions $\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \int g_{ij} \bar{\psi}^i \partial_t \psi^j + \text{int.}$

First order system should be treated like

Bosonic first order system — reminder.

$$\int\limits_{\substack{X(0)=X_0\\X(T)=X_T}} \mathcal{D}X\, e^{-\int \partial_t X^2 dt} = \int\limits_{\substack{X(0)=X_0\\X(T)=X_T}} \mathcal{D}p\mathcal{D}X\, e^{\int (ip\partial_t X - p^2) dt}$$

$\to$ no boundary conditions on p

So I treat fermions in the similar fasion

I impose boundary condition only on ψ

Altogether, the functional integral is a function of $\underline{\underline{X_0^i, \psi_0^i}}, X_T^i, \psi_T^i \in \Omega_X^\cdot \times \Omega_X^\cdot$, so I have an operator

$$\Omega_X^\cdot \to \Omega_X^\cdot$$

In functional integral approach we say that local observables are just local functionals of local fields

$$\int \mathcal{D}\varphi\, e^{\int_0^T \mathcal{L}(\varphi) dt}\, \mathcal{O}(\varphi)(t_1) = \quad 0 < t_1 < T$$

$$= e^{(T-t_1)H}\, \hat{\mathcal{O}}\, e^{t_1 H}$$

; suppose $T \to 0$

what are operators, corresp. to local observables?

$\hat{X}^i$ is just multiplication by X^i

~~bosonic~~ momentum

$$\widehat{g_{ij}\partial_t X^j} \Rightarrow \frac{\partial}{\partial x^i}$$

similarly, $\hat{\psi}^i$ is just multiplication by ψ^i

$$g_{ij}\hat{\bar{\psi}}^i \longrightarrow \frac{\partial}{\partial \psi^j}$$

being a momentum for ψ

<u>Reminder.</u> We observed it by studying p and π observables in instantonic approach:

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int \mathcal{D}p \; e^{\int p \frac{dx}{dt} - \epsilon p V(t_1)} \quad f(x)(t_2)$$

$t_1 > t_2$

 ϵ-jump

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Goal - show that operators announced $d + dW$ and $d^* + (dW)^*$ are actually Q and $\bar{Q}$

I want to show that

$$[d + \underline{dW}, x^i] = \delta_Q x^i \quad (a)$$

$$[d + dW, \psi^i] = \delta_Q \psi^i \quad (b)$$

$$[d + dW, \bar{\psi}^i] = \delta_{\bar{Q}} \bar{\psi}^i \quad (c)$$

a) is easy $d x^i = \psi^i$ $(+)$

b) also easy $0 = 0$

c) $\left\{ \psi^i \frac{\partial}{\partial x^i} + \psi^i \frac{\partial W}{\partial x^i}, \; g^{ij} \frac{\partial}{\partial \psi^j} \right\} =$

$$= g^{ij} \frac{\partial}{\partial x^j} + g^{ij} \frac{\partial W}{\partial x^j}$$

From $\delta_Q \psi^i = \underline{\underline{\partial_t x^i}} + F^i$ after integration over F

$$F^i = g^{ij} \frac{\partial W}{\partial x^j}$$

c) is proven.

Now, what about

$$\bar{Q} = d^* + (dW)^* = -g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial \psi^j} + g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial}{\partial \psi^j}$$

may be treated in the same way.

It is clear that

$$[\bar{Q}, \hat{x}^i] = -g^{ij}\frac{\partial}{\partial \psi^j} = -\hat{\bar{\psi}} \quad \text{as expected}$$

Similarly

$$[\bar{Q}, \psi^i] = -g^{ij}\frac{\partial}{\partial x^i} + g^{ij}\frac{\partial W}{\partial x^i}$$

$$\phantom{[\bar{Q}, \psi^i]} = -\partial_t \hat{x}^i + \hat{F}^i$$

Thus, Q and $\bar{Q}$ actually correspond to $d + dW$ and $d^* + (dW)^*$

$$D\Phi^i = \psi^i - \bar{\theta}\partial_t X^i + \bar{\theta}F^i - \bar{\theta}\theta\partial_t \psi^i \qquad (3)$$
$$\bar{D}\Phi^i = \bar{\psi}^i - \theta\partial_t X^i - \theta F^i - \bar{\theta}\theta\,\partial_t \bar{\psi}^i$$

Case 2 - nonconstant metric $g_{ij}(\Phi)$, what are new terms

$$\int d^2\theta\, g_{ij}(X + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F)\, D\Phi^i \bar{D}\Phi^i$$

$$\partial_K g_{ij} F^K \psi^i \bar{\psi}^j + \partial_K g_{ij}\,\psi^K \bar{\psi}^i (\partial_t X^i - F^i)$$

to get θ and

$$\partial_K g_{ij}\,\bar{\psi}^K \psi^i (\partial_t X^i - F^i)$$

$$+\ \partial_K \partial_l g_{ij}\,\psi^K \psi^l \bar{\psi}^i \psi^j \qquad (g_{ij}F^iF^j) \quad \text{we get}$$

by excluding F

$R_{ijKl}\,\psi^i \psi^j \bar{\psi}^K \bar{\psi}^l \longleftrightarrow$ How to see it?

$R \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma$, where Γ is

$$\Gamma = \frac{1}{2}\, g^{-1}(\partial g + \partial g - \partial g)$$

$R \sim \quad \partial g \cdot \partial g$ - terms and also $\partial^2 g$ terms.

$\partial_t X^i$ terms sum up into connection
Levi-Chivita connection.

$$S = \int g_{ij} \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial t} dt + \int g_{ij}\, \bar{\psi}^i \left(\partial_t \psi^j - \partial_t X^k \Gamma^i_{k\ell} \psi^\ell\right) dt +$$
$$+ \int R_{ijk\ell}\, \psi^i \psi^j \bar{\psi}^k \bar{\psi}^\ell dt + g^{ij} \frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j} + \frac{\partial^2 W}{\partial x^i \partial x^j}\psi^i \psi^j$$

New concept - Witten intex

Consider a space of states of an abstract TQM: V it is $\mathbb{Z}_2$-graded.

I assume that $H = \{Q, G\}$ has a discrete spectrum

$$I_T : \mathrm{Str}_V\, e^{-TH} = \mathrm{Tr}(-1)^p e^{-TH}. \qquad F\left(\bigcirc_T\right)$$

p is parity

theorem.

I_T does not depend on T

$$\frac{\partial I}{\partial T} = \mathrm{Tr}\,(-1)^p e^{-TH} (\underline{QG} + GQ) = 0$$
$\quad ①\qquad ②$

$$\mathrm{Tr} - Q(-1)^p e^{-TH} G = \qquad Q(-1)^p = -(-1)^p Q$$

$$= \mathrm{Tr} - (-1)^p e^{-TH} GQ = - ②$$

Important trick:

compare $T \to \infty$ and $T \to 0$ in the functional integral (in $N=1$ SQM)

$$T \to \infty \qquad \mathrm{str}_V\, e^{-TH} = \mathrm{str}\, e^{-TH} = \sum (-1)^k \dim H^k_d = \chi$$

Harmonic k forms, that are Ker of H

Euler number

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what happens in the opposite case $T \to 0$

Claim: when $T \to 0$ we localize to constant maps.

$$x^i(t) = x_0^i + x_n \exp \frac{2\pi i n t}{T}$$

$$\frac{dx}{dt} = \frac{2\pi i n}{T} x_n \exp \qquad \left(\frac{dx}{dt}\right)^2 \sim \frac{4\pi^2}{T^2} x_n^2$$

$$\int_0^T \left(\frac{dx}{dt}\right)^2 dt \sim \sum_n \frac{x_n^2}{T} \quad , \text{for } T \to 0$$

it is exp. vanishing - since it is in exponent.

Similar thing happens for fermions

Altogether, the functional integral (it may be shown) reduces to

$$\int dx_0 \, d\psi_0 \, \exp\left(R_{ijkl} \psi_0^i \psi_0^j \bar{\psi}_0^k \bar{\psi}_0^l \right) =$$

zero modes

= Gauss-Bonne formula for Euler number.

(We got functional integral proof)

One can prove many math. theorems using this idea (all index theory theorems)

Another consequence of $N=1$ SQM.

1) It is clear, that $d + dW = e^{-W} d \, e^{W}$, so

$$H_{d+dW}(X) = H_d$$

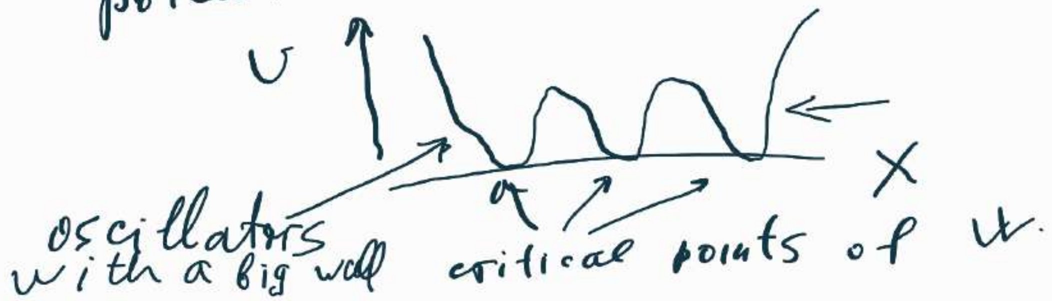
compact

2) At the same time $H_d \cong$ Harmonic forms,

and Harmonic forms correspond to potential $g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}$ + term linear in W

Consider rescaling $W \to \lambda W$, $\lambda \to +\infty$

potential looks like



Evaluate space of harmonic forms
its dimension is not greater then
the number of critical points
Each critical point has approximate
ground state with zero energy
and these states may only be lefted
due to tunnelling effect.
Tunneling can only deduce the
number of harmonic forms $= \dim H_d$

$\Downarrow$

Math. statement – number of critical
points of the function $W \geq$

$$\sum_k \dim H_d^k$$

Here we are coming again to
the framework of Morse theory –
– to be explained next time.