## Hodge theory and SQM

$$H = \{ d + dW, *(d+dW)* \}, \quad * : \Omega^p_X \to \Omega^{d-p}_X$$

$\dim X = d$

$$d = dx^\mu \frac{\partial}{\partial x^\mu} = \psi^\mu \frac{\partial}{\partial x^\mu}$$

$dx^\mu \equiv \psi^\mu$

$$*d* = g^{\mu\nu} \frac{\partial}{\partial \psi^\nu} \frac{\partial}{\partial x^\mu}$$

$$*dW* = g^{\mu\nu} \frac{\partial}{\partial \psi^\nu} \frac{\partial W}{\partial x^\mu} \to H = \Delta + \frac{\partial W}{\partial x^\nu} \frac{\partial W}{\partial x^\mu} g^{\mu\nu} + \ldots$$

Moral of having such $H$ — it has a discrete spectrum even on non-compact $X$ if $\frac{\partial W}{\partial x^\mu} \frac{\partial W}{\partial x^\nu} g^{\mu\nu} \to +\infty$ along noncompact directions, simplest example is $X = \mathbb{R}$

$$W = x^K$$

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In this case one can show that $H_{d+dW}$ on the space of dif.forms $\to 0$ at infinity coinside with so-called harmonic forms, that are annihilated by $H$.

$Q = d + dW$, and $Q^* = *(d+dW)*$

Really, let us diagonalize $H$.

Outside harmonic forms consider

$$Q\omega = 0 \Rightarrow$$

$$\omega = \frac{1}{H}(QQ^* + Q^*Q)\omega = Q\left(\frac{1}{H} Q^* \omega\right)$$

$$= \frac{1}{H} QQ^* \omega$$

so no cohomology outside harmonic forms

At the same time harmonic forms are closed $\to$ really

$$0 = \int \omega_h \{Q, Q^*\} \omega_h =$$

$$= \int \underset{\substack{\vee\!/ \\ 0}}{Q\omega_h * Q\omega_h} + \int \underset{\substack{\vee\!/ \\ 0}}{Q^*\omega_h * Q^*\omega_h}$$

$\Downarrow$

$Q\omega_h = 0$, so $Q$ acts by zero on harmonic forms

$H_Q \cong$ harmonic forms

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story becomes even more interesting for complex manifolds $X$

Really, consider $\mathbb{C}^n$ with the constant metric (hermitian).

Herm. means that only $g_{i\bar{j}} \neq 0$, $g_{ij} = g_{\bar{i}\bar{j}} = 0$

$i$ - holom. indexes, $\bar{i}$ - antiholomorphic.

$$\Delta = g^{i\bar{j}} \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^{\bar{j}}}$$ has 2 representations

Repr. 1: $\{d, d^*\} = \Delta$

Repr 2: $\bar{\partial} = d\bar{z}^{\bar{i}} \frac{\partial}{\partial \bar{z}^{\bar{i}}}$, $\bar{\partial}^+ = \overline{(* \bar{\partial} *)}$ ← complex conjugation

$$\Delta = \{\bar{\partial}^+, \bar{\partial}\}$$

And one may ask, when such relation holds for a curved space $X$?

$$\{d, d^*\} \overset{?}{=} \{\bar{\partial}^+, \bar{\partial}\}$$

when it is true for nonconstant metrics?

One may compute and find that (?)

holds when $\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}$ ← Kahler relation

and $\partial_{\bar{i}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{i}}$

metrics satisfying this relation are called Kahler.

For Kahler metrics there are 4 differentials

$\bar\partial, \partial, \bar\partial^+, \partial^+ \qquad d = \bar\partial + \partial$

they anticommute except

$\{\bar\partial, \bar\partial^+\} = \{\partial, \partial^+\} = \Delta$

In this way we identify

$H_{\bar\partial} \cong H_{\partial} \cong H_d$ by harmonic forms on compact manifolds $X$ where spectrum of $\Delta$ is discrete (Kahler)

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It turns out that even on non-compact manifolds such as $\mathbb{C}^n$ one can play the same game.

<u>Formulas</u>:

$W$ is holomorphic function on $X$

$\overline{W}$ is antihol. function on $X$

$$Q = \bar\partial + \partial W$$

$$\bar Q = \partial + \bar\partial \overline{W}$$

$$Q^+ = (\bar\partial + \partial W)^+ = \\ = *(\bar\partial + \partial W)* = *(\partial + \bar\partial \overline{W})*$$

$$\bar Q^+ = (\partial + \bar\partial \overline{W})^+ = *(\bar\partial + \partial W)*$$

One may compute that under Kahler condition they form the same package

only $\{Q, Q^+\} = \{\bar Q, \bar Q^+\} = H$

This was discovered only after physicists discover supersymmetry.

Topic - SQM, how it leads to Hodge theory

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These ideas - ideas of the superfields work also in theories of higher dimensions

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H corresponds to shifts in time
let us study symmetries of the action, generalizing shifts in time:
Assume we want to get the following algebra of sym:

$$\{Q, \bar{Q}\} = H \qquad (1)$$

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Remark: In higher dimensions we would like to get $Q_\alpha$ being spinors and algebra

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = \gamma^m_{\alpha\dot\beta} P_m$$

$\gamma$-matrixes

translations in space-time

Spinor $\times$ Spinor $\to$ Vector.

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In dim of space-time equal to 1, the only translation is translation in time, i.e. H.

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Introduce superspace with coordinates

$\theta, \bar\theta, t$

odd

$$Q = \frac{\partial}{\partial\theta} + \bar\theta\frac{\partial}{\partial t}, \quad \bar{Q} = \frac{\partial}{\partial\bar\theta} + \theta\frac{\partial}{\partial t}$$

$$\{Q, \bar{Q}\} = \frac{\partial}{\partial t}, \quad Q^2 = \bar{Q}^2 = 0$$

Comment:

There are D-operators, commuting with Q:

$$D = \frac{\partial}{\partial \theta} - \bar{\theta}\frac{\partial}{\partial t}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \theta\frac{\partial}{\partial t}$$

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Geometrical meaning of Q's and D's.

$\{Q, \bar{Q}\} = H$ may be considered as a lie superalgebra.

then $Q, \bar{Q}, H$ may be considered as left-invariant vector-fields on a Lie supergroup

$D, \bar{D}, H$ - may be considered as right invariant vector fields

$g \to h_L \, g \cdot k_R$, left and righ invariant vector fields correspond to obviously commuting actions on G so they (super) commute.

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Superfield - function on the superspace

look at invariants of the superfield

$$\Phi^i(t, \theta, \bar{\theta}) = \varphi^i_{(t)} + \theta \psi^i_{(t)} + \bar{\theta}\bar{\psi}^i_{(t)} + \theta\bar{\theta} F^i_{(t)}$$

components

even odd odd even

Action of supersymmetry on the superfield

$$Q\Phi^i = \left(\frac{\partial}{\partial \theta} + \bar{\theta}\frac{\partial}{\partial t}\right)\Phi^i = \psi^i + \bar{\theta}F^i + \bar{\theta}\frac{\partial \varphi^i}{\partial t} + \bar{\theta}\theta\frac{\partial \psi^i}{\partial t}$$

$$Q(\varphi^i) = \psi^i \qquad Q\bar{\psi}^i = \left(F^i + \frac{\partial \varphi^i}{\partial t}\right)$$

$$Q(\psi^i) = 0 \qquad QF = -\frac{\partial \psi^i}{\partial t}$$

$$Q^2(\bar{\psi}^i) = Q\left(F^i + \frac{\partial \varphi^i}{\partial t}\right) = -\frac{\partial \psi^i}{\partial t} + \frac{\partial}{\partial t}Q(\varphi^i) = 0$$

$$\bar{Q}(\varphi^i) = \bar{\Psi}^i \qquad \bar{Q}(\Psi^i) = \left(F + \frac{\partial \varphi^i}{\partial t}\right)$$

$$\bar{Q}(\bar{\Psi}^i) = 0 \qquad \bar{Q}F = -\frac{\partial \bar{\Psi}^i}{\partial t}$$

$$\{\bar{Q}, Q\}\varphi = \bar{Q}Q\varphi + Q\bar{Q}\varphi =$$

$$= \bar{Q}\Psi + Q\bar{\Psi} = 2 \cdot \frac{\partial \varphi}{\partial t}$$

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<u>Term 1</u>

$$\int d^2\theta\, dt\ \underline{\underline{G_{ij}\, D\varphi^i \bar{D}\varphi^j}}$$ – invariant under Q-symmetry

Reasons: a) $\{Q, D\} = 0$

b) $\int d^2\theta\, dt\ Q(\text{smth}) =$

$$= \int d^2\theta\, dt \cdot \left(\underline{\underline{\frac{\partial}{\partial \theta}}} + \theta \underline{\underline{\frac{\partial}{\partial t}}}\right)(\text{smth})$$

due to $\int d^2\theta$

vanishes due to $t$ integration.

<u>Term 2.</u>

$$\int d^2\theta\, dt\ W(\varphi)$$

The data in these terms is just the data in the system $\{d + dW, *(d+dW)*\}$

$W$ metric

more details

Term 1

$$D\varphi^i = \left(\frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial t}\right)\hat\varphi =$$

$$= \psi + \bar\theta F - \bar\theta\frac{\partial\varphi}{\partial t} - \bar\theta\theta\frac{\partial\psi}{\partial t}$$

$$\bar D\varphi^i =$$

$$= \bar\psi + \theta F - \theta\frac{\partial\varphi}{\partial t} + \theta\bar\theta\frac{\partial\bar\psi}{\partial t}$$

Assume that metric is constant

How to get $\theta\bar\theta$ terms for $\int d^2\theta$ integration?

$\bar\theta\theta F^2$, another source is to get them from ~ terms $\theta\bar\theta\left(\frac{\partial\varphi}{\partial t}\right)^2$

third way - from the dotted terms

$$\boxed{g_{ij}F^i F^j} + g_{ij}(\partial_t\varphi^i\partial_t\varphi^j) + \bar\psi^i\partial_t\psi^j g_{ij} \qquad \text{(term 1)}$$

Term 2.

$$\int d^2\theta\, W(\Phi)$$

How to get two $\theta$? $\quad W(\Phi) = \frac{\partial W}{\partial\varphi^i}F^i\theta\bar\theta + \frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j}\psi^i\bar\psi^j\theta\theta$

$$\boxed{\frac{\partial W}{\partial\varphi^i}F^i} + \frac{\partial^2 W}{\partial\varphi^i\partial\varphi^j}\psi^i\bar\psi^j$$

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Perform a Gaussian integral

over F $\longrightarrow$ $g^{ij} \dfrac{\partial W}{\partial \varphi^i} \dfrac{\partial W}{\partial \varphi^j}$

Altogether:

$$S = \int g_{ij}\, \partial_t \varphi^i \partial_t \varphi^j + g_{ij} \bar{\psi}^i \partial_t \psi^j + g^{ij} \frac{\partial W}{\partial \varphi^i} \frac{\partial W}{\partial \varphi^j} + \frac{\partial^2 W}{\partial \varphi^i \partial \varphi^j} \bar{\psi}^i \psi^j$$

Exactly the Term in potential

$* dW * dW$ corresponds to

$\{ d, * dW * \}$

here you also see second derivative in $W$

Moreover, what happens with supersymmetry?

In functorial QM

$$Q = d + dW = \psi^i \frac{\partial}{\partial \varphi^i} + \psi^i \frac{\partial W}{\partial \varphi^i}$$

How $Q$ was acting on fields?

$$Q(\varphi^i) = \psi^i,$$

$Q$ (on other fields) $\to$
$\to$ acts like predicted by SUSY. $\to$ we will see it tomorrow.