Derivation of quadratic relations from HTQM (in general) and morse theory as an example:

HTQM: V - space of states, Q - diff., G - unnormalized homotopy, $H = \{Q, G\}$

Use observables: $\varphi_i \in End(V)$, $[Q, \varphi_i] = 0$

Correlators as diff. forms on $\mathbb{R}^k_+$, defined as ...

"vacuum": $U_T = \exp TH$

assume that there is a limit $T \to \infty$

$\exists \lim\limits_{T\to\infty} U_T$. If such limit $\exists$ it is a projector $\lim\limits_{T_1, T_2 \to \infty} U_{T_1} U_{T_2} = \lim\limits_{T_1+T_2 \to \infty} U_{T_1+T_2}$

we project on kernel of H Vac

h_a: $H h_a = 0$ - vacuum states by definition

since $[Q, H] = 0$, space of vacuum states form a subcomplex in the complex V

$$\langle h_b, \varphi_{k+1} \ldots \underline{\exp(T_2 H + dT_2 G)}\, \varphi_2\, \underline{\exp(T_1 H + dT_1 G)}\, \varphi_1\, h_a \rangle = \underline{I}$$

$$\cap$$

$$\Omega^\bullet(\mathbb{R}_{+,k}) \otimes End(Vac)$$

I am going to study $\int\limits_{R_{+,k}} \underline{I}$ and find quadratic relations between these integrals

2 facts about $\underline{I}$

1) Factorization: let say $T_2 \to \infty$

$e^{T_2 H} \to$ projection

$$\underline{I}\Big(\underset{T_2}{\overline{\;|\;|\;|\quad|\;|\;}}\Big) \to \underline{I}\big(\;|\;|\;\big) \circ \underline{I}\big(\;|\;|\;\big)$$

composition in $End(Vac)$

2) Relative closedness

$\underline{I}$ is a dif. form $d^T \underline{I}$

d^T is a dif. with respect to all T_i

Easy case : assume that Q acts by zero on Vac

Then closedness follows from basic property of HTQM

$$d^T(\exp(TH + G\,dT)) = [Q, \exp(TH + G\,dT)]$$

in the easy case $d^T \underline{I} = 0$

Hard case – not that hard

$$d^T \underline{I} = [Q, \underline{I}]$$ here $\underline{I} \in \mathrm{End}(\mathrm{Vac})$ and Q are acting on Vac.

$$\int_{\substack{\text{boundary}\\ \text{of configuration}\\ \text{space}}} \underline{I} = 0 \to \text{easy case}$$

$[Q, \underline{I}]$ – hard case

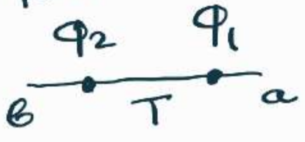
On the boundary of configuration space we may use factorization $\int \underline{I}\Big|_{\text{boundary}} = \int \underline{I} \circ \int \underline{I}$

origin of quadratic relations

Easy case

no T – not that interesting

$$\int_{\text{point}} \underline{I} = N_a^b(\varphi_1)$$ conf. space is a point



configuration space is formed by two copies of $\mathbb{R}_+$

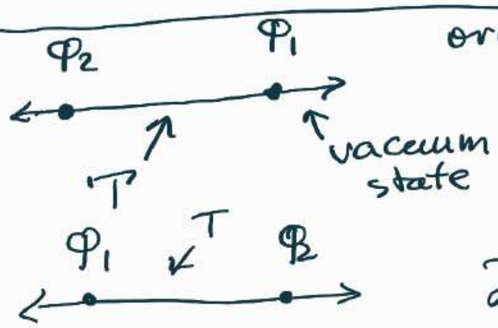
note, that $\underline{I}$ want to have only $T \to +\infty$ boundaries (where $\underline{I}$ have factorization)

$\mathbb{R}_+$ has two boundaries

what about boundary $T \to 0$

I forgot to put extra condition : $[\varphi_i, \varphi_j] = 0$

Under this extra condition contribution of $T \to 0$ boundary cancell between two copies of $\mathbb{R}_+$.



ordered points

$\mathbb{R}_+ \quad \partial R_+ = 0 \cup \infty$

$\partial \mathbb{R}_+ = 0 \cup \infty$

Rem. origin of condition $[\varphi_1, \varphi_2] = 0$ - actually graded commutator

From the def. theory of QFT we observed that

$$Q \to Q + \varphi_i \tilde{v}_i$$

$$[\varphi_i, \varphi_j] = 0 \longrightarrow Q^2_{def} = 0$$

$$0 = \int_{\partial R} I = \int_{\underset{\varphi_2 \ \infty \ \varphi_1}{\bullet\!-\!\bullet}} I + \int_{\underset{\varphi_1 \ \infty \ \varphi_2}{\bullet\!-\!\bullet}} I \underset{\uparrow \text{ factorization}}{=} \int_{\varphi_2} I \circ \int_{\varphi_1} I -$$

(∞ point)

$$\int_{\underset{\varphi_2 \ \infty \ \varphi_1}{\bullet\!-\!\bullet}} I_a^b(\varphi_2, \varphi_1) - \int_{\underset{\varphi_1 \ \infty \ \varphi_2}{\bullet\!-\!\bullet}} I_a^b(\varphi_1, \varphi_2) \underset{\uparrow \text{ factorization}}{=}$$

$$= I_c^b(\varphi_2) \cdot I_a^c(\varphi_1) - I_c^b(\varphi_1) \cdot I_a^c(\varphi_2) \to$$

$\to$ first quadratic relation $\quad I_a^b(\varphi) = N_a^b(\varphi)$

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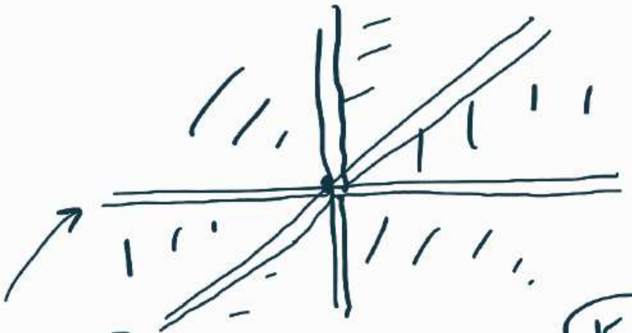
more complicated case

configuration space is $\mathbb{R}^2$ made out of

$6 \quad R^2_+$



6 cones.

each cone corresponds to some ordering of φ_i, s.

$(K=3)$

on this lines points coincide

there are boundaries: boundary configurations

it is easy to draw typical

boundary configuration

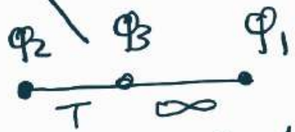
$$I^b_a(\varphi_3,\varphi_2,\varphi_1) \longrightarrow I^b_c(\varphi_3,\varphi_2)\cdot I^c_a(\varphi_1)$$

on the boundary of the type, depicted above

Now we take integral over boundary components

$$\int I^b_c(\varphi_3,\varphi_2) + \int I^b_c(\varphi_2,\varphi_3) \equiv$$

$$\equiv N^b_c(\varphi_3,\varphi_2)$$



So, alltogether we have relation between integrals, that looks like

$$N^b_c(\varphi_1,\varphi_2)\cdot N^c_a(\varphi_3) + \text{all permutations of } \varphi_i +$$

$$+ N^b_c(\varphi_1)\cdot N^c_a(\varphi_2,\varphi_3) + \text{all permutations} = 0$$

$$N(\varphi_1,\dots\varphi_K) = \int_{\text{over } \mathbb{R}^{K-1}} I(\underset{\text{nonordered}}{\varphi_1\dots\varphi_K}) \equiv \sum_{\sigma\in S_K} \int_{\text{over } \mathbb{R}_+^{K-1}} I(\underset{\text{over ordered set of observables}}{\varphi_{\sigma(1)},\dots,\varphi_{\sigma(K)}})$$

like

$$N(\varphi_1,\varphi_2) = \int_R I(\varphi_1,\varphi_2) = \int_{R_+} I(\varphi_1,\varphi_2) + \int_{R_+} I(\varphi_2,\varphi_1)$$

Consider the space

$$\mathbb{R}^K/\mathbb{R} = \text{config} = \mathbb{R}^{K-1}.$$

$$(t_1, \dots, t_K)\Big/_{(t_1+T, \dots t_K+T_K) \sim (t_1, \dots, t_K)}$$

at the same time, if all t_i are different, I can order them, and the same space config may be given like this

etc., with $T_i > 0$

So we get $K!$ configurations each isomorphic to $\mathbb{R}_+^{K-1}$

In the easy case these quadratic relations may be written as follows

Define universal observable

$$\varphi = \sum_i \varphi_i \tau_i \quad , \text{ such that parity of } (\varphi_i) + \text{parity of } (\tau_i) = 1 \pmod 2)$$

Define universal correlators

$$N_b^a(\tau) = \sum_K \frac{1}{K!} N_b^a(\underbrace{\varphi(\tau), \dots, \varphi(\tau)}_{K\text{-times}}))$$

then relations take the form

$$N_b^a(\tau) \cdot N_c^b(\tau) = 0 \quad \text{(combinatorical statement)}$$

Let us rerturn to hard case:

when $Q(\text{vac}) \neq 0$

Inspect our derivation

$$\int_{\partial\,\text{conf}} I_b^a = \int_{\text{conf}} dI_b^a = Q_c^a \underbrace{\int_{\text{conf}} I_b^c}_{N} - \underbrace{\int_{\text{conf}} I_c^a}_{N} Q_b^c$$

here Q_b^a is the action of Q on the vacuum subspace $Q h_a = Q_a^b h_b$

Alltogether we have an equation

$$N^a_b(\tau) N^b_c(\tau) = [Q,N]^a_c$$

Again this equation has a nice meaning:

$Q^a_b + N^a_b(\tau)$ squares to zero!

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up to now it was abstract.

Morse theory as an example:

$V = \Omega^\bullet_X$ $\quad H = L_V \quad$ V - was a Morse vector field

what is vacuum?

Vacuum is $\lim\limits_{\epsilon \to 0} \omega^{\epsilon}_{LT, a}$

what is a Lefshets thimple?

$\bullet \to$ A - critical point

consider all trajectories coming out of this point.

LT_A is a union of trajectories coming out of a critical point A.

Consider an example:

heart figure.

$W = -$ height

Blue figure is LT_A

Red figure is LT_B

Union of 2 two trajectories coming out of C is LT_C.

Point D is the LT_D

1) LT are invariant with respect to L_V
2) Differential d^x turns LT into LT

$d\omega^{\epsilon}_{LT_A} = \omega^{\epsilon}_{\partial LT_A} = \omega^{\epsilon}_{LT_C}$ - it works

3) How to see that

$$\lim_{T\to\infty} e^{TL_V}\omega \overset{?}{=} \sum_p \omega_{LT_p}$$

by inspection.

where ω_γ flows under the L_V flow:

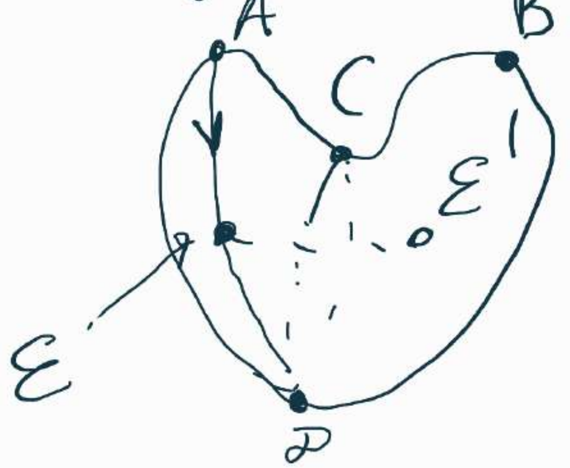
I will use that

$$e^{tL_V}\omega_Z^\epsilon = \omega^\epsilon_{e^{tV}Z}$$

On this figure we see, that

$$\omega_\gamma \longrightarrow \omega_{LH_C}$$

Number of curves passing through points



only one trajectory going from A to D through the point ε | Another enumerative question:

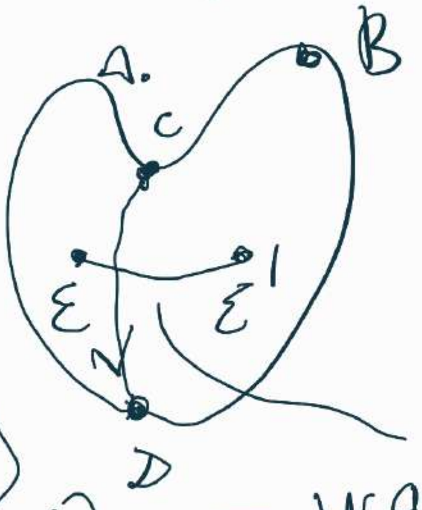
what would happen if I move a point ε to ε'

If $\varepsilon \to \varepsilon' \to$ no trajectories

The reason is still in the main formula.

$$\vec{I}^{\,a}_{b}(\varphi_1, \dots, \varphi_K)$$

we may ask $\varphi_i \to \varphi_i + Q(\tilde{\varphi}_i)$



$$\varphi = \omega_{\varepsilon}$$

$$\tilde{\varphi} = \omega_{\varepsilon\varepsilon'}$$

geo^m

Q - was just d^X

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since Vac is <u>not</u> Q-invariant

$\langle h_b, [Q, \tilde{\varphi}]\, h_a \rangle$ is not zero

actually, it is $\langle h_b, \tilde{\varphi}\, Q\, h_a \rangle =$

$\bullet \; \underline{\langle h_D \varphi_\varepsilon h_A \rangle} - \underline{\langle h_D \varphi_{\varepsilon'} h_A \rangle}$

$= \langle h_D [Q, \varphi_{\varepsilon\varepsilon'}] h_A \rangle =$ Algebra

$= \underline{\langle h_D \varphi_{\varepsilon\varepsilon'} Q h_A \rangle} =$

$= \langle h_D \varphi_{\varepsilon\varepsilon'} \underset{=}{h}_C \rangle \,.$

has enumerative meaning

– the number of curves coming out of C and intersecting the interval $\varepsilon\varepsilon'$.

Moral:

we may use not only closed φ_i, but also nonclosed φ_i → and still get relations between enumerative numbers.