1. Higher topological QFT is a unifying concept in mathematics & physics
2. Provide simple examples (from enumerative geometry) ← you may see them

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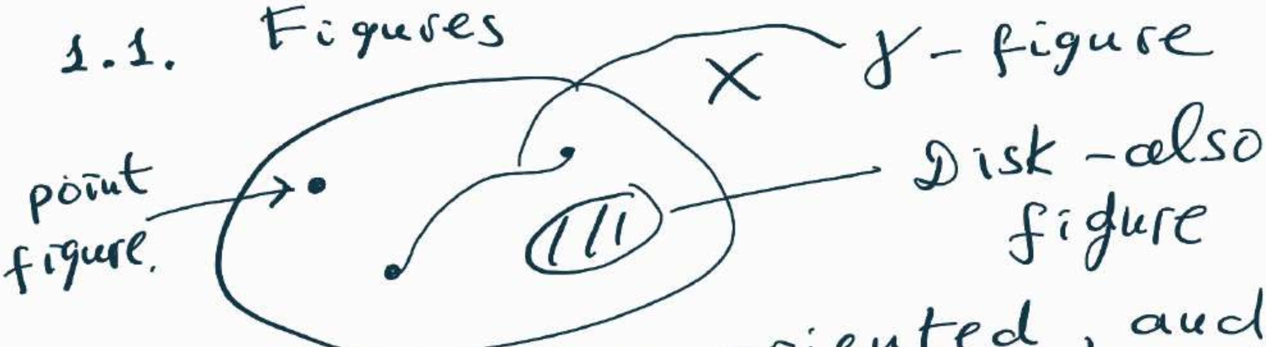
1. Figures and diff. forms

(Bott, Tu, Diff. forms and their applications in topology)

1.1. Figures



Assume that X is oriented, and figures are also oriented.

we say that γ_1 and γ_2 are oppositely oriented

we can generalize to disk.

D_1 and D_2 are oppositely oriented.

Construct an abelian group of Figures

$$FiG = \sum_{i=1}^{n} c_i \cdot FiG_i$$

c_i are just numbers
$c_i \in \mathbb{Z}$, or
$c_i \in \mathbb{Q}$, or
$c_i \in \mathbb{R}$, $c_i \in \mathbb{C}$

change of orientation

$$\overline{FiG} = -FiG$$

$$\overrightarrow{AB} = -\,\overleftarrow{AB}$$

Important concept — the boundary ∂

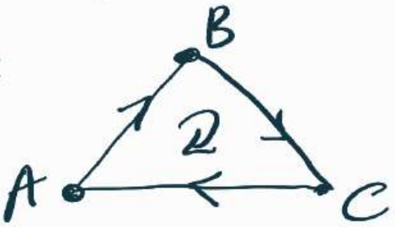
$$\partial(\overrightarrow{AB}) = B - A$$

as point

$$\partial(\overleftarrow{AB}) = A - B \quad \text{or}$$
$$= -\partial(\overrightarrow{AB}) = -(B-A) = A - B$$

The main property of the boundary operation is $\partial^2 = 0$

Example:

$\Delta =$ 

$$\partial(\Delta) = (AB) + (BC) + (CA)$$
$$\partial^2(\Delta) = B - A + C - B + A - C = 0 \quad !$$

Now, consider functions on X: Fun_X

$X \to \mathbb{R}$

Evaluation map:

$$\text{points} \times \mathrm{Fun}_X \to \mathbb{R}$$
$$(P, f) \longmapsto f(P)$$

This may be easily generalized to linear combinations of poins

$$(\text{zero-figures}, f) \longmapsto$$
$$\sum_i c_i P_i,\ f \longmapsto \sum_i c_i f(P_i)$$

I have a map between two linear spaces
This map is nongenerate = I may say that
functions are dual to zero figures

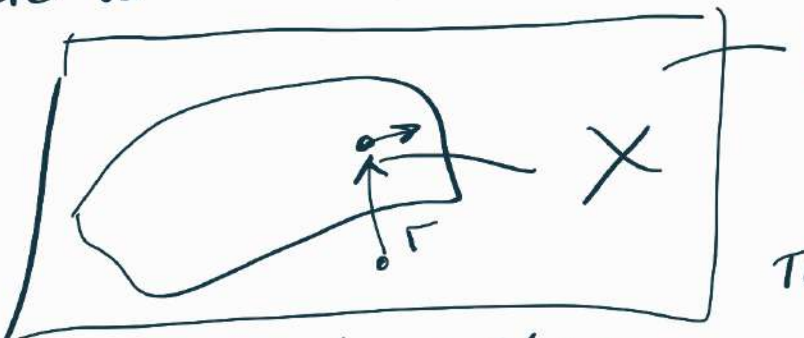
zero figures are a partical class of figures.
Looking for smth. dual to all figures
$ev(FiG, *) \longmapsto \mathbb{R}$, I would like to
have a linear map in both arguments
$* \to$ differential forms! One of the
definition of differential forms.

γ may be cut into small intervals (oriented)

would be differential form is just a map:

ω Taking tangent vector to a number
And we will assume this map to be linear: $\to$ cotangent vector depending on a point $\to$ 1-form.

Let us write it in coordinates:
Let x^μ be coordinates on X,
There is a tangent vector: $\frac{\partial \vec{r}}{\partial x^\mu}$ $\boxed{\text{in the lang of external geometry}}$



$\mathbb{R}^N$

$X: \vec{r}(x^1, \dots, x^d)$

Tangent vector
$\frac{\partial \vec{r}}{\partial x^\mu} = e_\mu$

In abstract setting
people forget $\vec{r}(x^1, \dots, x^d)$ and just say
that there is a vector field $\frac{\partial}{\partial x^\mu}$ understood
as derivative of the algebra of functions
$F(x^1, \dots, x^M) \longleftrightarrow$ algebraic point of view
Now, I have a dual object u^ν
would be
$u^\nu(e_\mu) = \delta^\nu_\mu$. Thus,
1 forms are $\omega_\nu(x) u^\nu$. Here I have
$(\omega_\nu(x) u^\nu, \vec{v}) = \omega_\nu(x_0) v^\nu_{x_0}$, if

$v = v^{\nu}_{x_0} \cdot e_{\nu}$ – these are differential forms

Now, what is evaluation map?

$(\omega_{\nu}(x)\, u^{\nu},$

$\gamma : \gamma(\tau)$

$\frac{\partial \vec{\gamma}}{\partial \tau}\Delta\tau$

$$\frac{\partial \vec{\gamma}}{\partial \tau} = \frac{\partial\, \vec{\gamma}(x(\tau))}{\partial \tau} = \underbrace{\frac{\partial \vec{\gamma}}{\partial x^{\mu}}}_{\vec{e}_{\mu}} \cdot \frac{\partial x^{\mu}}{\partial \tau}$$

$$\int \omega_{\nu}(x)\, \frac{\partial x^{\nu}}{\partial \tau}\, d\tau = \int_{\gamma} \omega$$

clearly, it is indep of parametrization $\gamma(\tau)$

It has multidim. generalization:

$\gamma \to$ Fig : $\vec{\gamma}(\tau_1, \dots, \tau_k)$ k-dim figure.

Dif. forms become

$$w = \omega_{\nu_1 \dots \nu_k}\, u^{\nu_1} \dots u^{\nu_k}$$

$$\int_{Fig} \omega = \int d\tau_1 \dots d\tau_k\ \omega_{\nu_1 \dots \nu_k}\, \frac{\partial x^{\nu_1}}{\partial \tau_1} \dots \frac{\partial x^{\nu_k}}{\partial \tau_k}$$

here $x^{\nu}(\tau_1, \dots, \tau_k)$ is a parametrized description of a figure.

Important remark. $\omega_{\nu_1 \dots \nu_k}$ should be antisymmetric in $\nu_1, \dots, \nu_k$.

(Known as an integral of the second kind)

Important remark:

geometrical diff. forms understood as linear functionals on figures are in one-to-one correspondence with Algebraic diff. forms understood as Fun $(\Pi T[1] X)$ – in simple terms

$= \omega_{\nu_1 \dots \nu_k}(x) \cdot \psi^{\nu_1} \dots \psi^{\nu_k}$, here

ψ^ν are odd coordinates on the fiber.

Moreover, we may define the operation ∂^* ?

$ev(Fig, \partial^* \omega) \equiv ev(\partial Fig, \omega)$

it is a definition of ∂^*!

From this definition it follows that $(\partial^*)^2 = 0 \to$ we just conjugate twice!

Great theorem – 7 authors theorem (Arnold, Math. methods in classical mechanics, part II)

$\boxed{\partial^* = d^{DR}}$, where $d = \psi^\mu \frac{\partial}{\partial x^\mu}$.

Newton, Leibnitz, Green, Cochy, Gauss, Ostrogradsky, Stocks

$\int_a^b F' dx = F(b) - F(a)$

$\updownarrow$

$d^{DR} F' = ev(F', \partial [a b])$

$\quad\quad\quad\quad\quad\quad\; ''$

$ev(\partial^* F, [a, b])$

Recall, that Ω_X form an algebra so we may expect an algebra on Figures

Poincare duality.

$\int_X \omega_1 \cdot \omega_2$ – pairing on differential forms, smth-like this on figures

Namely:

Step 1. I would like to associate to a figure Z (called chain) a diff. form ω_Z with a property

$$\int_Z \omega = \int_X \omega_Z \cdot \omega$$

this multiplication is understood as a multiplication in the ring $\mathrm{Fun}(X,Y)$

Step 2.

Ask: what is $\omega_{Z_1} \cdot \omega_{Z_2} \leftrightarrow ?$

Let me start with simplest examples

$X = \mathbb{R}$; figures on $\mathbb{R}$ - points and intervals

Let me start with an interval

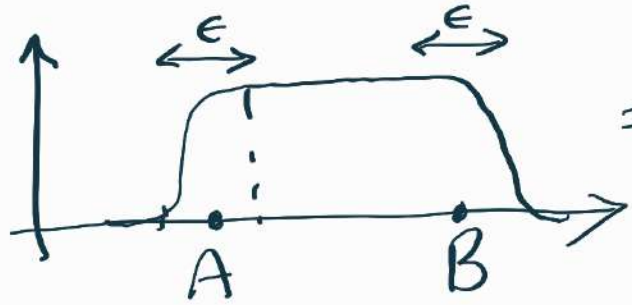
$I = (A, B) \to$ 1-form

$$\int_R \omega_I \cdot \omega^{(1)} = \int_{[A,B]} \omega^{(1)}$$

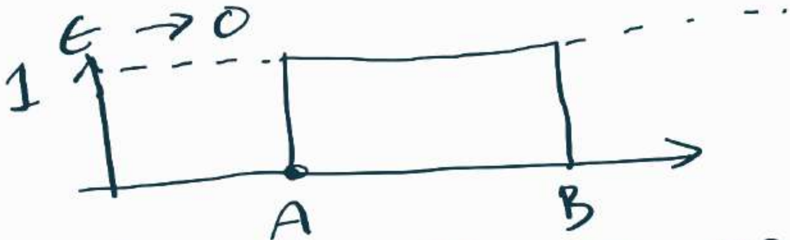
Interestingly, I cannot find such ω_I in the space of smooth forms! instead, I can have an appoximate form ω_I^ϵ, such that ω_I^ϵ is smooth and

$$\lim_{\epsilon \to 0} \int_R \omega_I^\epsilon \cdot \omega^{(1)} = \int_{A,B} \omega^{(1)}$$

; this ω_I^ϵ - should be a it should be just a zero form, i.e. function.



$= \omega_I^\epsilon$ is smoothed $-\theta(x-B)+\theta(x-A)$

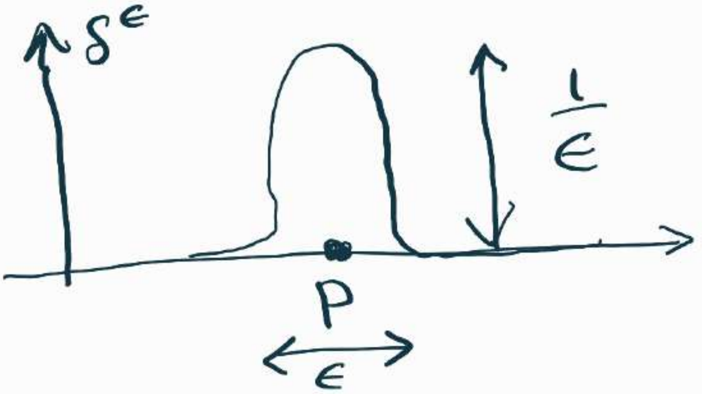


problem solved.

Another figure – a point P.

$$\lim_{\epsilon\to 0}\int_R \underset{\uparrow}{\omega_P^\epsilon}\cdot \underset{\uparrow}{f} = f(P) = \int_P f$$

1-form   function, i.e. a zero form.



$\omega_P^\epsilon = \delta^\epsilon \cdot \underset{\psi}{\underset{\shortparallel}{dx}}$

$$\boxed{\psi = \psi\frac{\partial}{\partial x}x = \underline{\underline{dx}}}$$

$d\omega_Z^\epsilon$, from the main theorem: I assume that X is compact

$$\underline{\underline{\int_X d\omega_Z^\epsilon\cdot\omega}} = -\int_X \omega_Z^\epsilon\, d\omega =$$

$$= -\int_Z d\omega + O(\epsilon) =$$

$$= -\int_{\partial Z}\omega + O(\epsilon) =$$

$$= -\int_X \underline{\underline{\omega_{\partial Z}\cdot\omega}} + O(\epsilon)$$

$$d\omega_z^\epsilon = -\omega_{\partial z}^\epsilon + O(\epsilon)$$

In dim $X = 1$ we constructed a ϵ-map between figures and dif. forms

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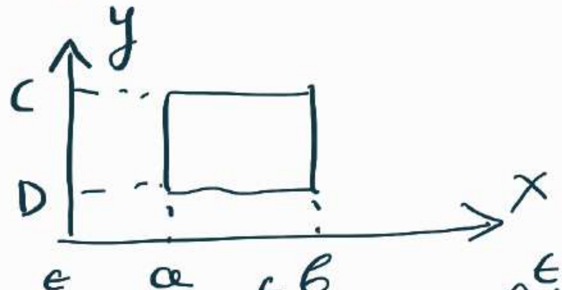
Let us go to dim $X = 2$

$p \to$ point $\leftrightarrow$

$$p \to \delta^\epsilon(x-x_0)\,\delta^\epsilon(y-y_0)\,dx\,dy = \omega_p^\epsilon$$

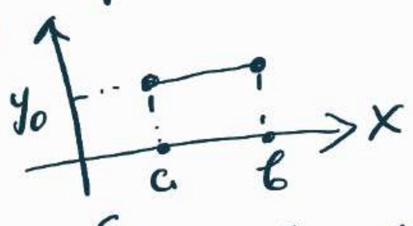
$$\int_{\mathbb{R}^2} \omega_p^\epsilon \cdot f = f(x_0, y_0) + O(\epsilon)$$

Z being a square



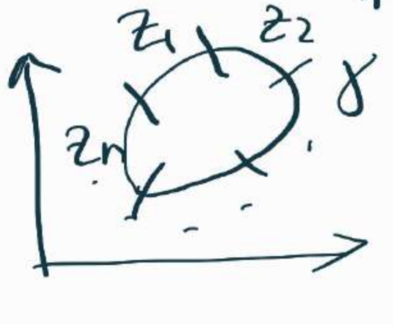
$$\omega_Z^\epsilon = \left(\theta^\epsilon(x-b) - \theta^\epsilon(x-a)\right)\left(\theta^\epsilon(y-c) - \theta^\epsilon(y-D)\right)$$

what corresponds to an interval



$$\int_{\mathbb{R}^2} \omega^{(1)}\, \omega_I^\epsilon = \int_I \omega^{(1)}$$

$$\omega_I^\epsilon = dy \cdot \delta^\epsilon(y-y_0) \cdot \left[\theta^\epsilon(x-b) - \theta^\epsilon(x-a)\right] \cdot + O(\epsilon)$$



$\omega_\gamma^\epsilon =$

split it into regions,

not

$\omega^\epsilon_{z_1+z_2} = \omega^\epsilon_{z_1} + \omega^\epsilon_{z_2}$ – so we can split into regions, an it is enough to get $\omega^\epsilon_{z_1}$

here we may take a local coordinates where z_1 corresponds to $y_1 = 0$

and write $y = \varphi(x)$

$$\omega^\epsilon_y = \delta^\epsilon(y - \varphi(x))\, dy \cdot \left(\theta^\epsilon(x - x_1) - \theta^\epsilon(x - x_2)\right) + O(\epsilon)$$

And so on in higher dimensions:

Thus, we constructed a map FIG $\to$ DIF. FORMS

$$z \to \omega^\epsilon_z$$

$$\omega^{\epsilon_1}_{z_1} \cdot \omega^{\epsilon_2}_{z_2} = \omega_{?}$$

$$\omega^{\epsilon_1}_{z_1} \cdot \omega^{\epsilon_2}_{z_2} = \omega^\epsilon_{z_1 \cap z_2} + O(\epsilon)$$

Important issue

deg in ψ in the l.h. side is $\operatorname{codim} Z_1 + \operatorname{codim} Z_2$

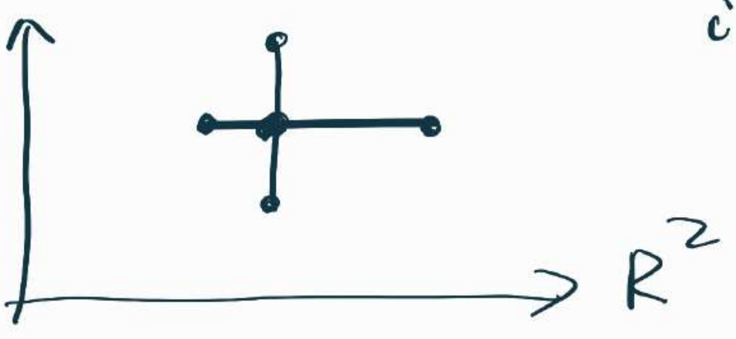
so for fixed ϵ if $\operatorname{codim} Z_1 + \operatorname{codim} Z_2 > X$ we have zero!

$$\omega_p^{\epsilon} = \delta_p^{\epsilon} \cdot dx$$

$$\omega_{p'}^{\epsilon} = \delta_{p'}^{\epsilon}\, dx.$$

$$\omega_p^{\epsilon} \cdot \omega_{p'}^{\epsilon} = 0$$ for all p and p'

$$\omega_p^{\epsilon_1} \cdot \omega_{\underline{I}}^{\epsilon_2} = \begin{cases} \omega_p + O(\epsilon) & \text{if } p \in \underline{I} \\ 0 & \text{if } p \notin \underline{I} \end{cases}$$



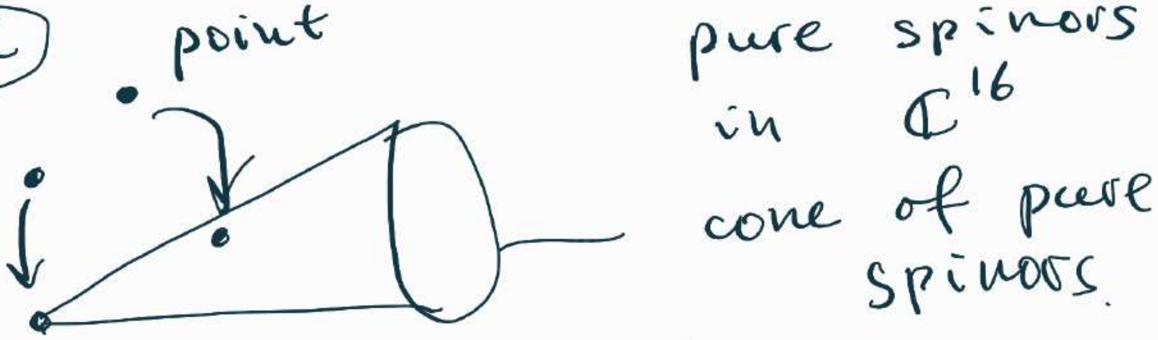
in $\mathbb{R}^2$

$$\omega_{\underline{I}_1}^{\epsilon} \cdot \omega_{\underline{I}_2}^{\epsilon} = \; = \omega_{\underline{I}_1 \cap \underline{I}_2}^{\epsilon} + O(\epsilon)$$

(v)

This (almost) completes the classical dictionary between algebra & geometry

$\boxed{\text{For future}}$



Naively, you get nothing

however, if point with a regular point of a cone you get a space whose coordinate ring is an External algebra $\cong$ purely odd dim. space.

moreover, if you intersect a point with a tip of the cone $\to$ polarizations of SYM theory!

(happens in derived geometry)