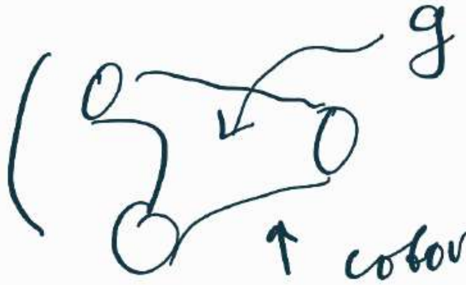
Higher topological QFT

with the simplest examples
HTQM.

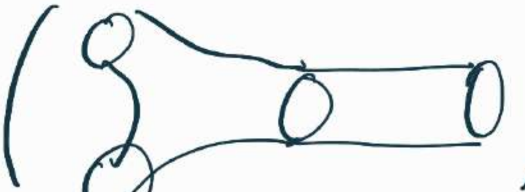
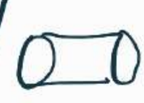
Functor from category of geom. enhanced (enriched) cobordisms to cat of vector spaces.

$F\Big($ 

$\Big) \in \mathrm{Hom}(V_{in}, V_{out}) \otimes \mathrm{Fun}(\text{geom})$

↑ cobordism

geometrical enhancement

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

For each geometry g_1

$$ev(F, g_1) \in \mathrm{Hom}(V_{in}, V_{out})$$

evaluation

Simplest example here is just 1-dim smooth cobordisms, geom. enhancement – just metric

$F\Big($ 

$\Big) = F\Big($ 

$\Big) \circ F\Big($ 

$\Big)$

In particular, if we assume inv. w.r.t. diff. of the cobordism

g_1/diff → length of the interval

$$F(T_1+T_2) = F(T_1)\circ F(T_2)$$

$$F(T) = \exp(T\hat{H})$$ ← call it <u>Hamiltonian</u>.

Math. ⇒ supermathematics

manifolds → Q-supermanifolds

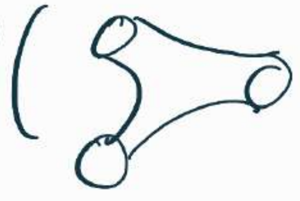
vector spaces → complexes

Fun(geom) → dif. forms on geom ≅ Fun(T[1]geom)

Fun(T[1]geom) may be considered as a Q-manifold with the hom. vector field d^{DR}

($\underline{T[1]\mathrm{geom}}$, d^{DR}) is a model of the new manifold that I will call Top.

From the modern point of view

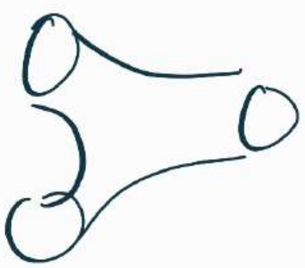
$F\Big($  $\Big) \in \mathrm{Hom}(V_{in}, V_{out}) \otimes \Omega^\bullet\,\mathrm{geom}$

Moreover, from the modern point of view,

(!) V_{in}, V_{out} should be considered as complexes: $Q: V_{in} \to V_{in}$, $Q^2 = 0$
(name Q comes from math-physics)

$\boxed{\text{Extra AXIOM.}}$ Higher topological QFT is a

<u>CLOSED</u> Functor from geom. enriched cobordisms $\to$ <u>complexes</u> of vector spaces

F () $\in \mathrm{Hom}(V_{in}, V_{out}) \otimes \Omega^{\bullet}_{geom}$

and $(d + Q) F = 0$ (!)

closeness axiom.

We will see that it is also natural from the point of view of higher mathematics (take into acount homotopies) Abstract bla-bla-bla.

Sencible math is

Abstract bla-bla-bla +

\+ interesting examples.

Consider simplest example - QM:

$$F\left(\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \right) = \hat{F}(T, dT)$$

here I move from

Fun (lenghts) →

→ Dif. forms of the lengths

Functorrality!

$$F(T_1+T_2, dT_1+dT_2) = F(T_1, dT_1) \circ F(T_2, dT_2)$$

closeness (closedness)

$$(d^T + Q)\, F(T, dT) = 0$$

Q is a dif. in V

$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$

V is ~~a~~ complexe.

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solution

$G: V \to V$

G is odd

$G^2 = 0$

$$\exp\{\underline{\underline{d^T + Q}}, \underline{\underline{G \cdot T}}\} =$$

$$= \boxed{\exp(H\cdot T - G\, dT)} =$$

$$\boxed{H = \{Q, G\}}$$

$$= \exp(TH)(1 - G\, dT)$$

Here I use that

$[H, G] = 0$, really:

$$[G, \{G, Q\}] = 0 \text{ due to } G^2 = 0.$$

Thus, topological QM is given by a pair. Q and G, such that

$Q^2 = 0$, $G^2 = 0$

For any diff. $\mathcal{D}$ Lemma.

$[\mathcal{D}, e^{\{\mathcal{D}, R\}}] = 0$ - for closedness

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but I also need Functoriality:

Explain Lemma.

1. On conceptual level in hom. algebra there are no simbols like $=$, the only simbol is $\simeq$

John Baez slogan

$a = b$ is either trival or wrong

if a is b it is trivial

if a is not b it is wrong

$a \underset{\gamma}{\sim} b$ — this is the only reasonable formula on mathematics → it reads a is equivalent to b in eq. relation γ!

homotopical view on mathematics.

Sience called algebraic topology → we turn hom. considerations into some kind of an algebraic structure.

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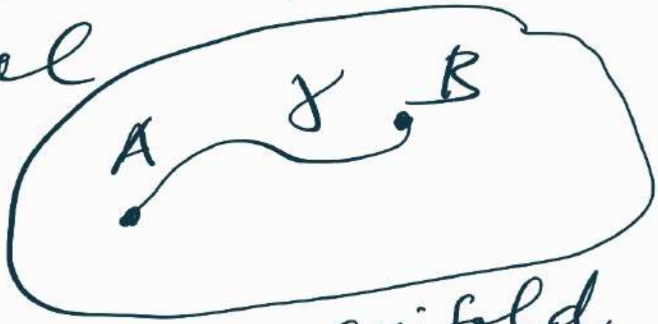
1) we formally may add and subtract objects (take their formal linear combinations)
2) We understand $a \underset{\gamma}{\sim} b$ as

$$a - b = D(c)$$

we replace homotopy by differential

Example!



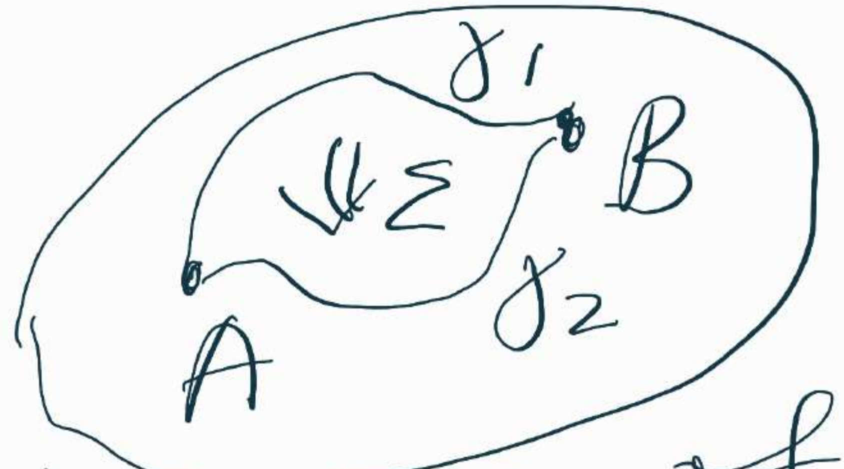
Two points on the manifold that

belong to the same connected component may be connected by a path γ.

Point is a Figure, path is also a Figure

$$B - A = \partial(\gamma)$$

boundary operation

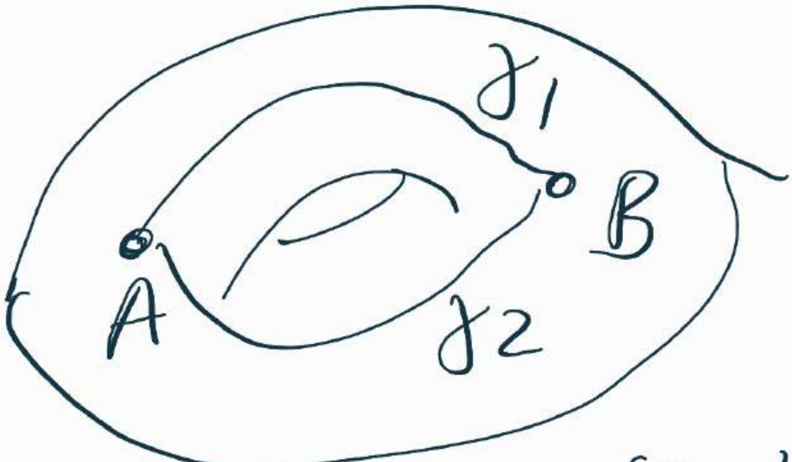


Continuation of example: it could be that γ_1 may be deformed into γ_2, so there is Σ: $\partial(\Sigma) = \gamma_2 - \gamma_1$

in particular

$$\partial^2(\Sigma) = \partial(\gamma_2) - \partial(\gamma_1) =$$
$$= (B-A) - (B-A) =$$
$$= 0$$

Follows from the universal principle $\partial^2 = 0$. In this case we say that two homotopies γ_1 and γ_2 are equivalents. It may happen, that they are not equivalent: Example



γ_1 and γ_2 are <u>not</u>

equivalent (or not homotopical to each other) $\rightarrow$ build alg. topology:

$$\partial(\gamma_1-\gamma_2)=0$$

but $\gamma_1-\gamma_2 \neq \partial\Sigma$

It means that

$$\gamma_1-\gamma_2 \in H_\partial \ (\text{1-cycles})$$

etc.

Now back to concept understanding of the formula:

$$[\mathcal{D}, e^{\{\mathcal{D}, R\}}]=0$$

$\{\mathcal{D}, R\}$ is homotopical $\underline{\underline{\text{zero}}}$

$e^{\{\mathcal{D}, R\}}$ is homotopical

e^{0} i.e. 1

$e^{\{\mathcal{D}, R\}}$ is homotopical 1.

And surely,

$[\mathcal{D}, 1] = 0.$

Now, algebraic proof:

$$[\mathcal{D}, e^{\{\mathcal{D}, R\}}] =$$

$$= [\underbrace{\mathcal{D}, 1}_{0} + \underline{\underline{\{\mathcal{D}, R\}}} + \frac{1}{2}\{\mathcal{D}, R\}\{\mathcal{D}, R\} + \ldots$$

$[\mathcal{D}, \{\mathcal{D}, R\}] = 0$ it follows from $\mathcal{D}^2 = 0$ - but

may be pronounced as $\mathcal{D}$ of hom. zero is zero

$$[\mathcal{D}, \frac{1}{2}\{\mathcal{D}, R\}\{\mathcal{D}, R\}] = \quad \leftarrow \text{Leibnitz rule}$$

$$= \frac{1}{2}\underbrace{[\mathcal{D}, \{\mathcal{D}, R\}]}_{0}\{\mathcal{D}, R\} + \{\mathcal{D}, R\}\underbrace{[\mathcal{D}, \{\mathcal{D}, R\}]}_{0} =$$

and so on $\xrightarrow{=0}$ Lemma is proved.

Question of Tim

Are there other solutions to a system:

$$\begin{cases} F(T_1+T_2, dT_1+dT_2) = \\ = F(T_1, dT_1) \circ F(T_2, dT_2) \\ dF + [Q, F] = 0 \end{cases} \quad \text{(TQM)}$$

I proposed the solution

$$F = e^{T\{Q,G\} - dT\,G}$$

Are there other solutions?

Examples of my solution:

Example 1.

consider a manifold X,

take V to be $\Omega^\bullet_X$

take Q to be d_X.

Equip X with a metric g,

take $G = d_X^* = - * d_X *$

Then $H = \{d_X, d_X^*\} = \Delta_X$

<u>Example 2.</u> Again, consider manifold X, and V to be $\Omega^\bullet_X$, and Q to be d_X.

But now equip X with a vector field $v \in \mathrm{Vect}(X)$ and take

$$G = \iota_v$$

(if $\Omega^\bullet_X = \mathrm{Fun}(x^\mu, \psi^\mu)$, then $\iota_v = v^\mu \frac{\partial}{\partial \psi^\mu}$)

Then

$$H = \{d_X, \iota_v\} = \mathcal{L}_v$$

Lie derivative

Cartan formula

moral of examples.

Theory considered as physical (with $H = \Delta$) is a subsector of HTQM.

It means that if I restrict myself to subsector $V_0 \subset V$ of diff. forms of degree 0 I have a physical system describing a free particle moving on X!

It works for a free particle, what about potential?

Let me conjugate d by e^W

W – is just some function on X

$d^W = e^{-W} d e^W = d + dW$, clearly, $(d^W)^2 = 0$

(by the way, d^W is known as Witten's differential). W was a standard notation for so-called superpotential had no relation to Witten.

$$Q = d^W, \quad G = * d^W *, \text{ so}$$

$$H_W = \{ d^W, * d^W * \}$$

$$H_W = -\Delta + \{ dW, * dW * \} + \text{terms linear in } W$$

$dW \to \underline{\underline{\psi^i}} \frac{\partial W}{\partial x^i}$

$* dW * \to g^{ij} \frac{\partial W}{\partial x^j} \underline{\underline{\frac{\partial}{\partial \psi^i}}}$

metric on X

together $\{dW, * dW *\} = g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}$

$-\Delta + g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}$

potential appears.

two extra terms

$\{d, * dW *\} \stackrel{?}{=}$ (for simplicity consider constant g^{ij})

$= \{ \underline{\underline{\psi^i}} \underline{\underline{\frac{\partial}{\partial x^i}}}, g^{ij} \underline{\underline{\frac{\partial W}{\partial x^j}}} \underline{\underline{\frac{\partial}{\partial \psi^i}}} \} =$

$= g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial}{\partial x^i} + g^{ij} \frac{\partial^2 W}{\partial x^i \partial x^j} \psi^i \frac{\partial}{\partial \psi^j}$

Similary, I will get a term from

$\{* d *, dW\} \to$

$\{ g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \psi^j}, \psi^i \frac{\partial W}{\partial x^i} \} =$

$= g^{ij} \frac{\partial^2 W}{\partial x^i \partial x^j} \psi^i \frac{\partial}{\partial \psi^j} +$

$+ g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial}{\partial x^j}$

Altogether

$$V = \frac{\partial W}{\partial x^i}\frac{\partial W}{\partial x^j} g^{ij} + W'' \psi \frac{\partial}{\partial \psi} +$$

$$+ \text{ probably } g^{ij}\frac{\partial W}{\partial x^i}\frac{\partial}{\partial x^j}$$

Now,

Δ_W is clearly Hermitean

it can not involve terms like

$u^i(x)\frac{\partial}{\partial x^i}$ – they are

antihermitean :- have to cancell out

so potential is

$$\|dW\|^2_g + (\partial^2 W)\cdot \psi\frac{\partial}{\partial \psi} \text{ terms}$$

Once again, if we restric to Ω^0_X → no dependence on ψ → we will get a potential →

Moreover: H_W preserves Ω^0_X

so we have a Q.M. with potential V as a subsystem of HTQM !

conjecture →

→ (?) → could it always be like this, i.e. <u>QFT</u> ⊂ <u>HTQFT</u> ?

Interestingly, for QM it is always true!

Suppose, we have a QM with space of states V (image of a boundary point due to Functor: $F(\bullet) = V$).
Consider a new QM that would turn out to be a HTQM:

$$V^{HT} = \mathbb{C}^{1|1} \otimes V$$

QM had a Hamiltonian $H : V \to V$

$Q = \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix}$ a block representation due to $V^{HT} = \mathbb{C}^2 \otimes V$ structure

$G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then

$\{Q, G\} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}$

so on states of the form $\begin{pmatrix} v \\ 0 \end{pmatrix}$
$\{Q, G\}$ equals H. Embedding of QM into HTQM.

Equivalent writing:

Consider an odd number c

due to historical reasons

$$V^{HTQM} = \mathbb{C}[c] \otimes V$$

$$\mathbb{C}[c] = \mathrm{span}(1, c)$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \text{ are even}$$

$$\begin{pmatrix} 0 \\ v_2 \end{pmatrix} \text{ are odd.}$$

Operators, that map even to odd and odd to even are odd.

Equivalent writing:

Consider an odd number c

due to historical reasons

$$V^{HTQM} = \mathbb{C}[c] \otimes V$$

$$\mathbb{C}[c] = \mathrm{span}(1, c)$$

$$\boxed{Q = cH, \quad G = \frac{\partial}{\partial c}}$$

$$\{Q, G\} = \{cH, \frac{\partial}{\partial c}\} = H$$

For future; in dim 2 CQFT these formulas would be generalized to

$$Q = \int c T_{matter} + \frac{1}{2} c T^{ghost}$$

$$G = b$$

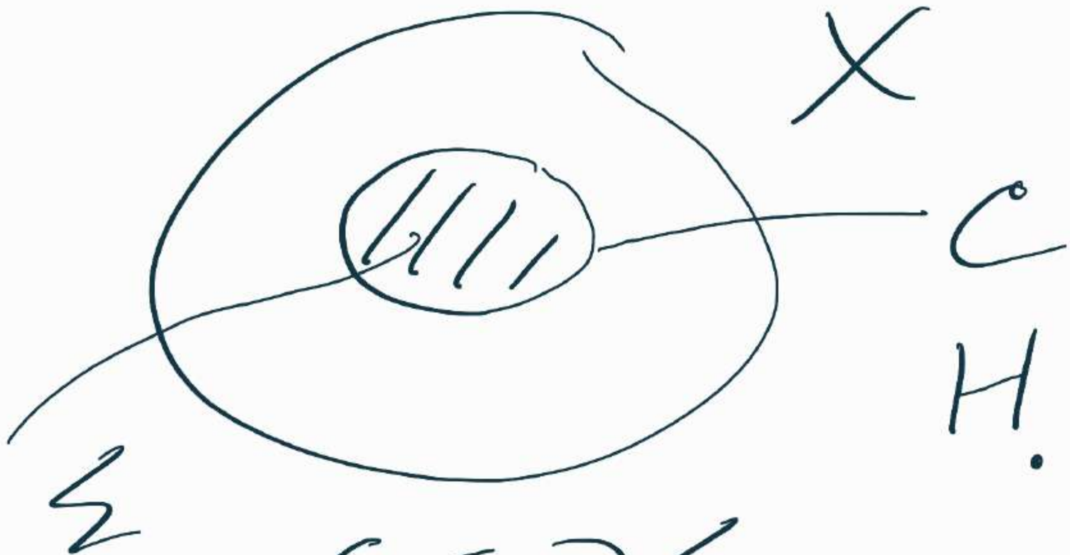
$$G = \frac{\partial}{\partial c} \qquad \int \mathcal{D}b \mathcal{D}c \, e^{\int_0^T b \frac{dc}{dt} dT} = K(c_1, c_0)$$

$c(0) = c_0$
$c(T) = c_1$

$b \to \frac{\partial}{\partial c_0}$ just like in

$$\int \mathcal{D}x \mathcal{D}p \, e^{\int_0^T p \frac{dx}{dt} dT}$$

where $p \to \frac{\partial}{\partial x}$



$H_\cdot(C) = 0$

$C = \partial \Sigma$

In Higher theory we trace all homotopies
we are not saying that $c = 0$,
we rather say, $c = \partial \Sigma$

$H \neq 0$, $H = \{Q, G\}$

Example from enumerative geometry

linking of two cycles

Linking is:

$$\boxed{c_1 = \partial \Sigma_1} \quad |||$$

$$\mathrm{Link}(c_1, c_2) = \Sigma_1 \cap c_2 \ldots$$