From instantonic theories to higher topological QFT.

kind of Functional integral examples $\longleftrightarrow$ functorial QFT

$\hookrightarrow$ is an actual definition of QFT

(!) main definition

M.Q. integral for trajectories of the vector field:

$$\frac{dx^i}{dt} - v^i(x) = 0$$

$$S = \int \underline{p_i \frac{dx^i}{dt} + \pi_i \frac{d\psi^i}{dt}} \underbrace{- p_i v^i(x) - \pi_i \frac{\partial v^i}{\partial x^j}\psi^j}_{(MQ)}$$

M.Q. action

From the point of view of Hamiltonian mechanics we see

bosonic configuration space with coord x^i (just X), and fermionic (odd) conf. space with coord ψ^i

(all together, conf. space is $T[1]X$)

Space of states are just functions on the configuration space: $\mathrm{Fun}(T[1]X) = \Omega^{\cdot} X$

$$[0,T] \xrightarrow{\gamma} T^*(T[1]X)$$

phase space (momenta p_i and π_i)

$$\int\limits_{\substack{x(0)=x_0 \;\; \psi(0)=\psi_0 \\ x(T)=x_1 \;\; \psi(T)=\psi_1}} \mathcal{D}\gamma \, \exp\frac{i}{\hbar}(S_{MQ}) = K(x_0,\psi_0; x_1,\psi_1; T)$$

K is an integral kernel of
the operator $e^{\frac{i}{\hbar}T\hat{H}}$

$\hat{H}$ is expected to be $p_i v^i(x) + \pi_i \frac{\partial v^i}{\partial x^j}\psi^j$

---

we would like to understand it as
an operator $\Omega^\bullet_X \to \Omega^\bullet_X$

From previous examples we saw that

$p_i \Leftrightarrow \frac{\partial}{\partial x^i}$, so $\hat{H} = v^i(x)\frac{\partial}{\partial x^i} +$

$\pi_i \Leftrightarrow \frac{\partial}{\partial \psi^i}$ $\quad + \frac{\partial v^i(x)}{\partial x^j}\psi^j \frac{\partial}{\partial \psi^i}$

just a Lie derivative acting
on diff. forms

Really; $\underline{\underline{\mathcal{L}_v = \{d, \iota_v\}}}$ Cartan formula

$d = \psi^i \frac{\partial}{\partial x^i}$, formula that
$\iota_v = v^j \frac{\partial}{\partial \psi^j}$ ← mathem. do not like.

$\{d, \iota_v\} =$
$= v^j \frac{\partial}{\partial x^j} + \left[\frac{\partial}{\partial x^i}, v^j\right]\psi^i \frac{\partial}{\partial \psi^j}$ $\quad \underline{\underline{d\, x^j}} = \psi^i \frac{\partial}{\partial x^i} x^j =$
$\underline{\underline{= \psi^j}}$

I actually get the hamiltonian
of the action (MQ)

$\mathcal{L}_v$ is a (quantized) Hamiltonian on the
superspace $T^*(T[1]X)$

$\hbar = 1$

$K(x_0, \psi_0, x_1, \psi_1; T) \Leftrightarrow \exp T\mathcal{L}_v$

F. I. formula $\qquad$ functorial QFT

what is $\exp T\mathcal{L}_v$?

it is a diff. γ_T

$$(\exp T L_V)\, \omega = \gamma_T^* \omega$$

γ_T is a diffeomorphism !!!

Moreover, this theory may be interpreted in <u>enumerative geometry</u>.

Enumerative geometry is a part of geometry that studies numbers of figures of special type

Simplest questions in E. geometry are: $\mathbb{C}P^1 = S^2$

Number of intersection points of 2 geodesics

answer is $\underline{\underline{2}}$.

Or. how many solutions the equation $P_n(z) = 0$ has $\to$ intersection of two curves $y = 0$, z arbitrary and $y = P_n(z)$

n is the degree of pol. P_n

Answer is n.

Another question in enumerative geometry is intersection of homology classes

C_1 - cycle in $[C_1]$

C_2 - cycle in $[C_2]$

Question: # of points in $C_1 \cap C_2$

Easy theorem - this number is indep. of the respresentative $c \subset [C]$

How enumerative geometry is related to M.-Q. quantum mechanics?

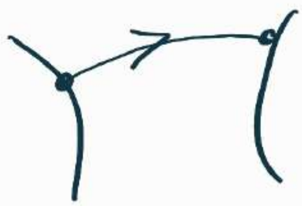
Let ω_0^{ϵ} be a dif. form that is smoothening of the δ^{ϵ}-form $\delta^{\epsilon}_{c_0}$ (it can also be written in terms of M.-Q. integral)

Consider the following number

$$\int_{\substack{dx_0 dx_1 \\ d\psi_0 d\psi_1}} K(x_0, \psi_0, x_1, \psi_1; T)\, \omega_0^{\epsilon_0}(x_0, \psi_0)\, \omega_1^{\epsilon_1}(x_1, \psi_1) = f(c_0, c_1; T)$$

Statement 1.

$f(c_0, c_1, T)$ computes the number of trajectories starting at c_0 and ending at c_1 in a time T.



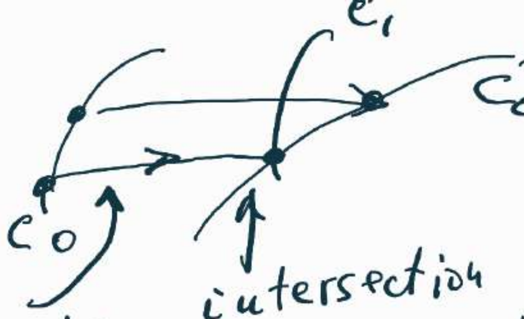
Two proofs:

Pr. a) From M.Q. integral instantons are just all trajectories of $v \Leftrightarrow \delta$ trajectories

$\lim\limits_{\epsilon_0, \epsilon_1 \to 0} \int \delta^{\epsilon_0}_{c_0} \delta^{\epsilon_1}_{c_1} \to$ computes exactly trajectories connecting c_0 and c_1.

trajectories

Pr b) $K(x_0, \psi_0, x_1, \psi_1)$ is an integral kernel of the operation γ_T^* on dif. forms $\to \gamma_T$ on figures.

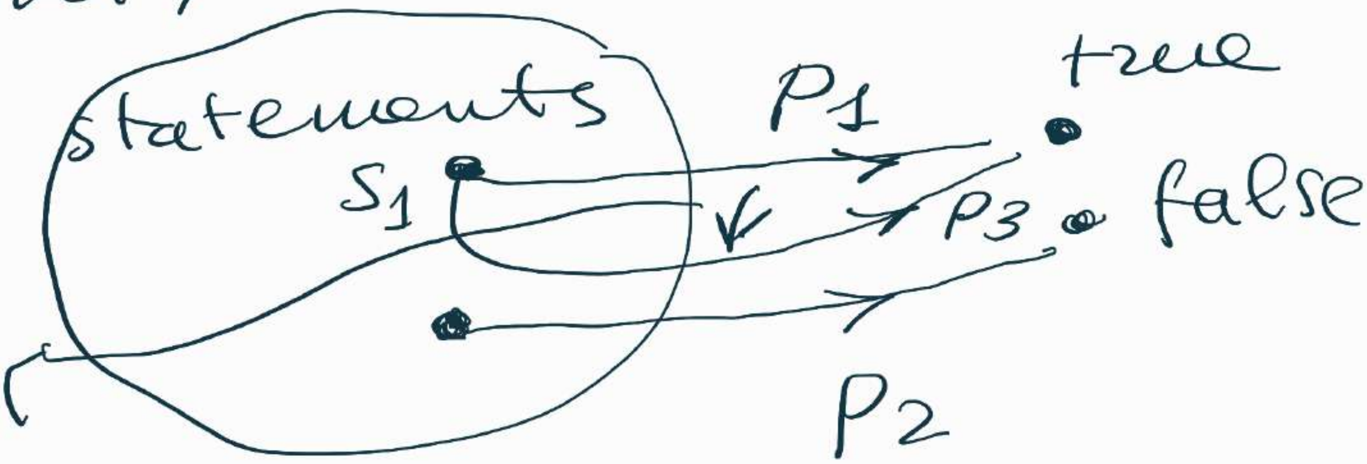


$c_0^2 = \gamma_T(c_0)$, and then we study intersection of $\gamma_T(c_0) \cap C_1$

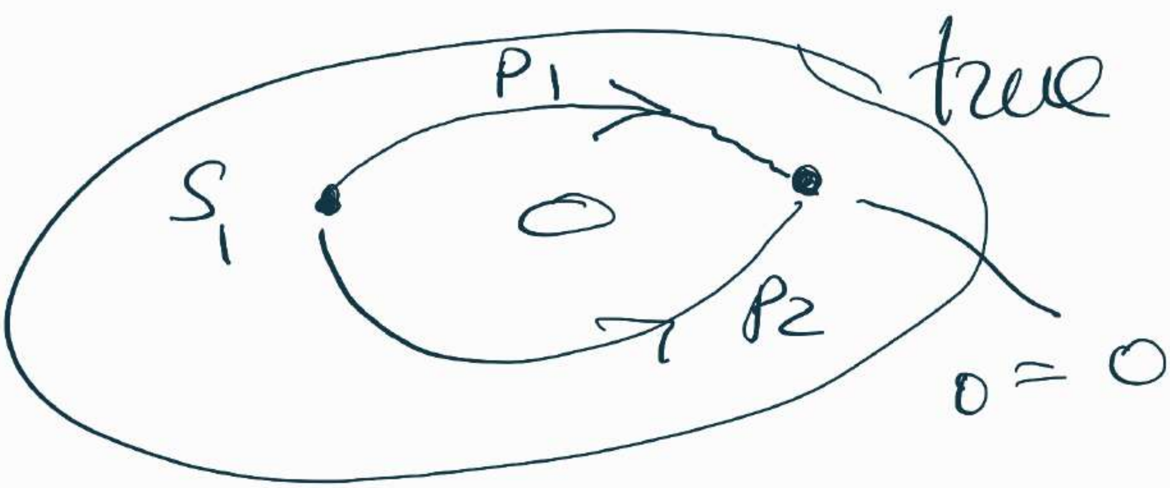
(functorial proof of the statement)

Comparing M.-Q. and Functorial proofs of the same statement is a good exercise in understanding equiv. between M.Q. & Funct. views.

Higher math due to Baez



homotopy



Topology in the space of proofs

Sometimes

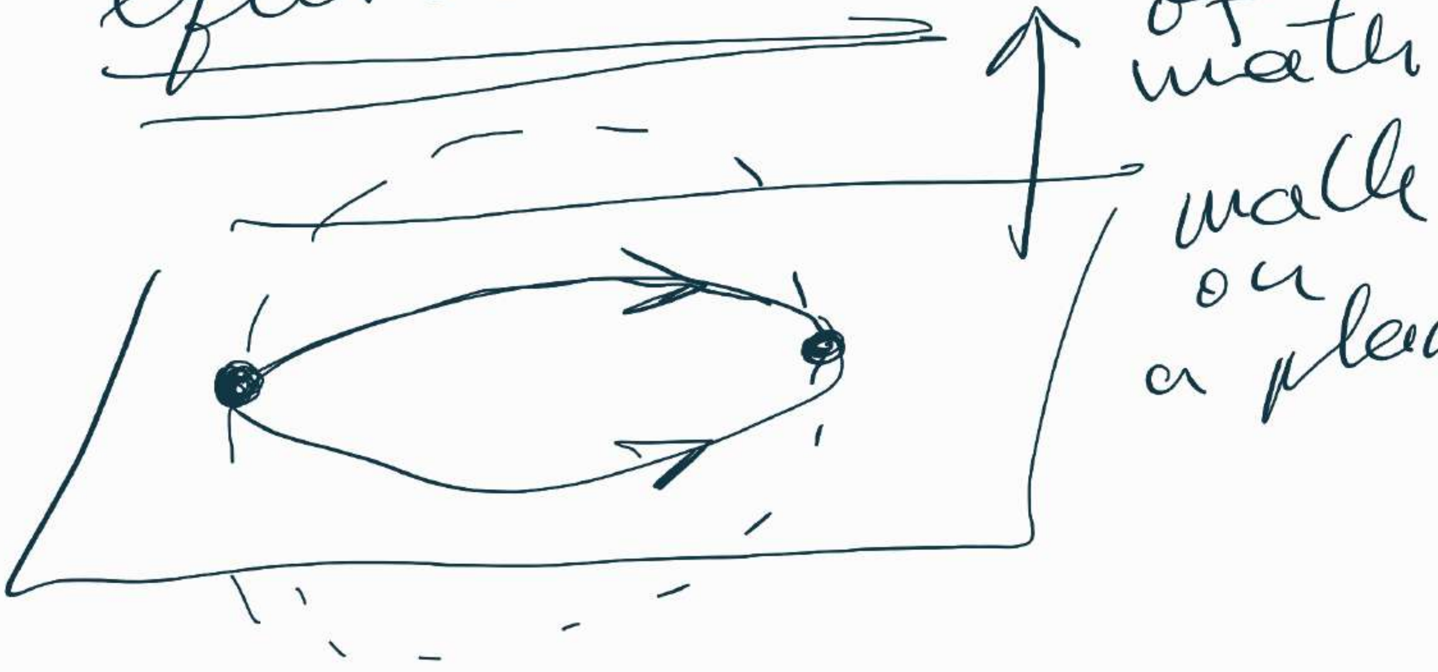
$$\begin{tikzcd} S \arrow[r, bend left, "p_1"] \arrow[r, bend right, "p_2"'] \arrow[r, phantom, "\Downarrow"] & \text{true} \end{tikzcd}$$

(BAEZ)

proofs p_1 & p_2 may be homotopically the same

Two proofs a) & b) of a statement in enumerative geometry are the same $\boxed{(!)}$ since $K(\varphi_0, X_0, \varphi_1, X_1) =$ integral kernel of γ_1^*

Development of math is to make ~~all~~ "different proofs equivalent."



We have an example - we will study it in more details.

Question 1.
Does the number
$$C_0 \cap \gamma_T(C_1) \text{ depend on } T?$$

Question 2.
Does the number $C_0 \cap \gamma_T(C_1)$ depend on v?

Let us study question 1.

proof from functorial Q.M. that answer is NO.

Argument $N_T = C_0 \cap \gamma_T(C_1) = \int_X \omega_1 e^{TLv} \omega_0$

$$\frac{\partial}{\partial T} N_T = \int_X \omega_1 \, L_V \, e^{T L_V} \omega_0$$

Now important trick.

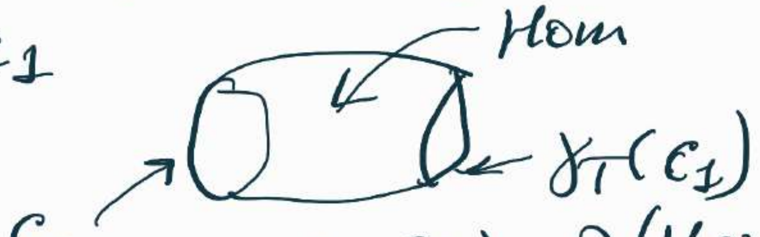
we use that $L_V = \{d, \iota_V\}$

$$\frac{\partial N_T}{\partial T} = \int_X \omega_1 \{d, \iota_V\} e^{T L_V} \omega_0 \qquad (1)$$

Now, $[d, e^{T L_V}] = 0$, if $d\omega_0 = 0$ and $d\omega_1 = 0$, from (1) it follows that $\frac{\partial N_T}{\partial T} = 0$ in particular, we got an interesting statement in enumerative geometry

$$C_0 \cap \gamma_T(C_1) = C_0 \cap C_1$$

In enumerative geometry people would not be impressed: actually, $\gamma_T(C_1)$ is a homotopy of C_1



$$C_1 - \gamma_T(C_1) = \partial(\mathrm{Hom})$$

Corollary, $C_0 \cap \gamma_T(C_1)$ is indep. of V

Let us change the game.

En. geometry studies figures, not mess. cycles.

what would happen if we go to figures (in en. geometry called chains).

Simplest example of a chain — interval [A,B]

$\partial[A,B] = B - A$

point point

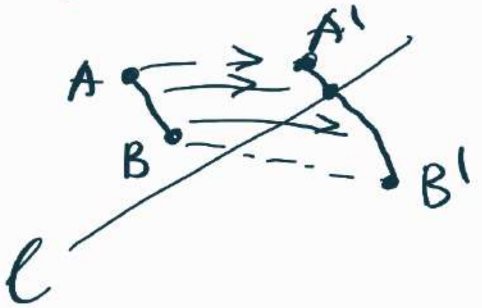
Question: what can we say about

$[A,B] \cap C$ — ?

It is clear that it is not homotopical invariant

Example:

$(AB) \xrightarrow[\text{def.}]{} (A'B')$



It is clear

$(AB) \cap l = 0$

$(A'B') \cap l = 1$

So it is not homotopical invariant

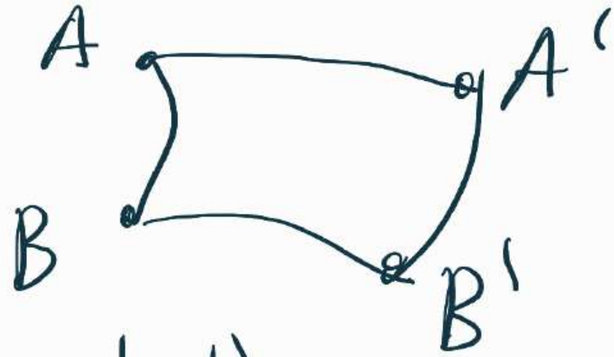
? Can we improve the situation,

what is invariant ?

Bad things happen, when B occurs on a line → jump.

Actually, when we had a cycle

$C - C' = \partial(\ \ )$

Now 

$(AB) - (A'B')$ is not a boundary!

The boundary is

$$\left[(AB) - (A'B') - (AA') - (BB')\right] \cap \ell = 0$$

$$(AB) \cap \ell - (A'B') \cap \ell = (AA') \cap \ell + (BB') \cap \ell \quad (*)$$

<u>This was enumerative geometry</u>

Now, we would like to study

$$C_1 \cap \gamma_T Z_0$$

Z_0 is a notation for a chain

$$\partial Z_0 \neq \emptyset$$

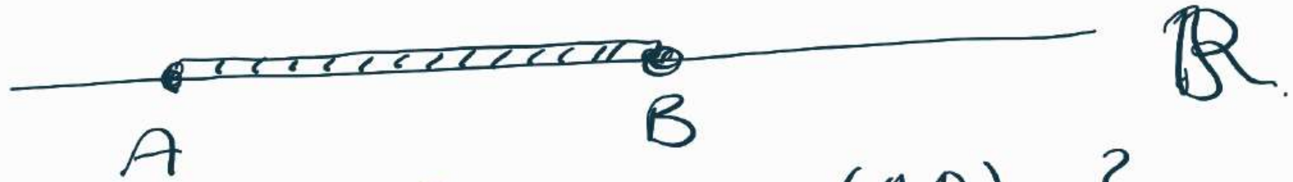
we may still write

$\int \omega_1 e^{TL_V} \omega_0^{\epsilon}$ but now

$d\omega_0 \neq 0$

$$\boxed{d(\delta_Z) = \delta_{\partial Z} \quad (!)}$$

Example of (!)



what is δ on ~~a~~ (AB) ?

it is a $\theta(x-B) - \theta(x-A)$

Then, $d\big(\theta(x-B) - \theta(x-A)\big) =$

$$= \delta(x-B) - \delta(x-A) =$$

$$= \delta_B - \delta_A$$

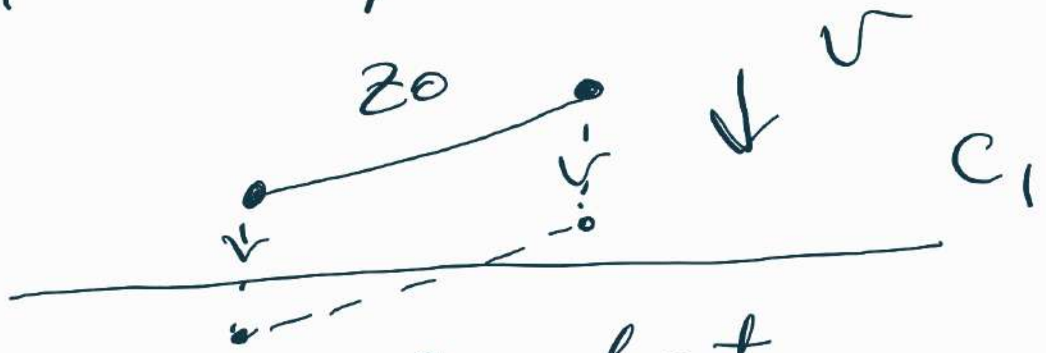
This is an illustration of (!)

To get more details → book Bott–Tu, dif. forms and their application in topology?

Now we may address questions (1) and (2) to

$C_1 \cap \gamma_T(z_0)$

Ex: Let z_0 be an interval on the plane



$z_0 \cap C_1 = 0$ but

$C_1 \cap \gamma_T(z_0) = 1$, so answer depend on T!

---

what happens with the argument?

$$\frac{\partial N_T}{\partial T} = \int \omega_1 L_v e^{TL_v} \omega_0$$

$$L_v = \{d, i_v\}$$

$$\frac{\partial N_T}{\partial T} = \int \omega_1 \{d, i_v\} e^{TL_v} \omega_0 = \int_X \omega_1 i_v e^{TL_v} \underbrace{d\omega_0}_{\neq 0}$$

$$d\omega_0 = \delta_{\partial Z_0}$$

$$= \int_X \omega_1 \, e^{TL_v} i_v \, \delta_{\partial Z_0} =$$

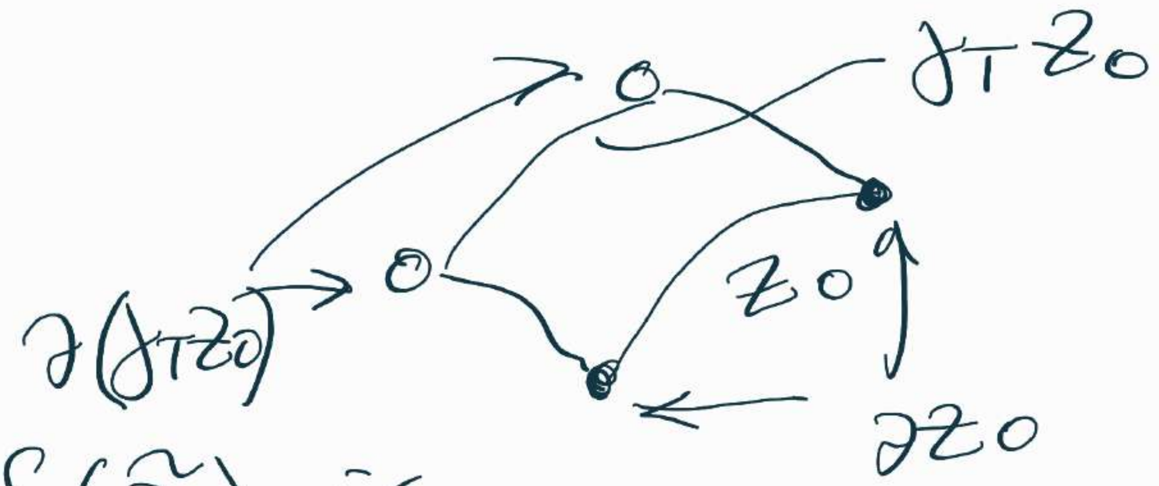
what is the geometrical meaning of this integral ?

$$= \int_X \omega_1 \; i_v \; \delta_{\partial(\gamma_T Z_0)} \quad (2)$$

Do I understand geometry of the formula (2): L.H.S $\frac{\partial N_T}{\partial T}$

$N_{T_2} - N_{T_1} \qquad T_2 = T_1 + \Delta T$

$$\Delta T \frac{\partial N_T}{\partial T} = \Delta T \int_X \omega_1 i_v \delta \underline{\underline{(\partial \gamma_T Z_0)}} \quad (3)$$



Now

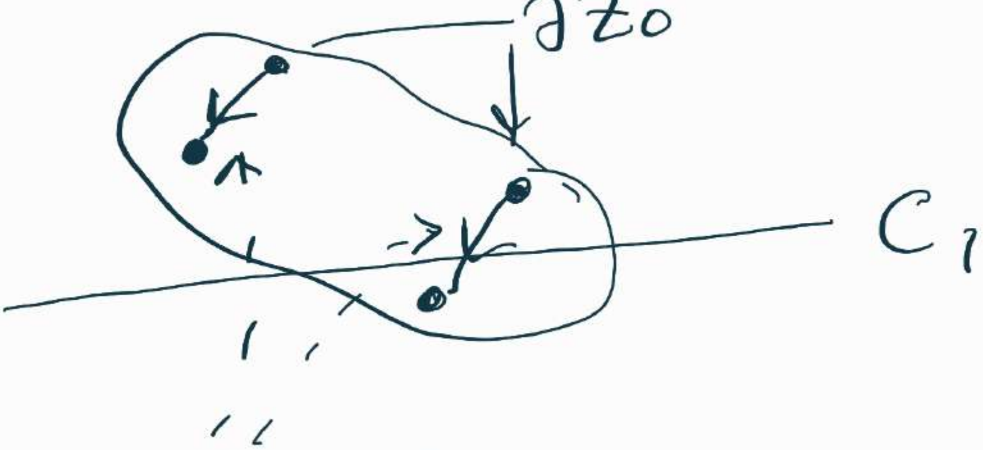
$\Delta T \, i_v \, \delta(\tilde{Z})$ is

$\delta(i_{v\Delta T}\cdot \partial Z_0)$

$(i_{v\Delta T}\cdot Z)$ it is a figure

$\Delta N =$

$= \int \omega_1 \cdot \delta(i_{v\Delta T}\,\partial Z_0)$



$\Phi = i_{v\Delta T}\cdot \partial Z_0$, so

$\boxed{\Delta N = \Phi \cap C_1}$ that is what we expected geometrically.

Goals of the exersize

1. Show that $c_i \to Z_i$ and N depends on T

2. Show that algebraic manipulations with functorial formula $\int_{W_1} e^{TL_V} \omega_0$ correspond

X

to clear geometrical pictures.