New approach for F.I.

$$\int \mathcal{D}x\, e^{\frac{i}{\hbar}\int\left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - \frac{k}{2}x^2\right)dt}$$

it was important that space of fields X had linear structure. It determined the measure $\mathcal{D}x$: $\mathcal{D}(x+h) = \mathcal{D}x$

concept of sum — depends on linear structure

In class. mechanics.

$$S = \int g_{ij}(x)\,\frac{dx^i}{dt}\frac{dx^j}{dt}\,dt$$ — square of velocity

on some manifold X without any linear structure $X =$ 

How to do F.I. in this case? and in similar cases?

Idea is the following

1.) S as critical values of

$$\int p_j \frac{dx^j}{dt} + p_i p_j G^{ij}(x)$$

2) study integrals like

$$\int \mathcal{D}x\,\mathcal{D}p\, \exp i\int\left[p_j \frac{dx^j}{dt} + p_i p_j G^{ij}(x)\right] \quad (*)$$

3) consider here as a perturbation and observe that $\int \mathcal{D}p\, \exp i\int p_j \frac{dx^j}{dt}$ is "δ-function" on solutions to equation $\frac{dx^j}{dt} = 0$

4) Generalize.

(*) is what I am going to get at the end.

Step 1. Mathai–Quillen represent. for δ-function

Step 2. Understand

$$\int \mathcal{D}x\mathcal{D}p \exp \int p_j \frac{dx^j}{dt} dt \quad x(t_1)\dots x(t_k) \quad p(\tau_1)\dots p(\tau_k)$$

through finite dimensional integrals over ("instantons")

Instantonic theories of Frenkel, Nekrasov & L.

Step 3. Take it as a (math) definition of a functional integral.

Step 1 $\quad X$ - space

$\omega_1, \dots, \omega_k$ - dif. forms on X

$\omega = \omega(x, \psi) = \omega_{j_1\dots j_p}(x)\psi^{j_1}\dots\psi^{j_p}$

Let N be a space of zeroes of functions F_a. We will assume that N is finite dimensional

How to write

$\int_N \omega_1 \dots \omega_k$ in terms of integral over X?

Idea - it would not be an integral over X, it would be an integral over some supermanifold

$Y = \mathcal{E}[1] \times X \quad \mathcal{E}$ is the space of equations

$\mathcal{E}[1]$ - means that it is an odd space with coordinates $\pi^a \leftarrow$ odd.

Explicite formula

$$\int_N \omega_1 \dots \omega_k = \int_{MQ} d\pi^a dp^a dx^i d\psi^i \exp\left(ip^a F_a + i\pi^a \frac{\partial F_a}{\partial x^j}\psi^j\right) \omega_1(x,\psi)\dots\omega_k(x,\psi) \qquad (**)$$

Idea of the proof:

1) consider $\exp\left(i p^a F_a + i\pi^a \frac{\partial F_a}{\partial x^j}\psi^j - \epsilon p^a p^b G_{ab}\right)$

2) Take integral over dp^a variables

get $\frac{1}{(\sqrt{\epsilon})^{\cdot}}\, e^{-\frac{1}{\epsilon} F_a F_b G^{ab}}$, $\quad G^{ab} = G^{-1} \to$ inverse

number of equations

look as smth. distributed along zeroes of F_a

$F_a = 0$

$$** \approx \int dx\, d\psi\; \omega_1(y,\psi_y)\dots\omega_n(y,\psi_y) + \epsilon\text{-corrections} \to 0$$

when $\epsilon \to 0$

<u>Example:</u>

$n=1$, $\omega_1 = f(x)$, a also equals to 1.

$$\int dx\, d\psi\, \frac{1}{\sqrt{\epsilon}} \int d\pi\; e^{-\frac{F^2}{\epsilon}} f(x)\, e^{\pi F' \psi} =$$

$$\int dx\, \frac{e^{-F^2/\epsilon}}{\sqrt{\epsilon}}\, f(x)\cdot F'(x)$$; suppose that $F(y)=0$, y is zero of F

$$\frac{1}{\sqrt{\epsilon}} \int dx\; e^{-\frac{1}{\epsilon}(F'(y))^2 (x-y)^2} f(x)\left(F'(y) + (x-y)F'' + \dots\right)$$

$$= \frac{\sqrt{\epsilon}}{\sqrt{\epsilon}} \left(\frac{1}{F'(y)} f(y)\, F'(y) + O(\epsilon)\right) =$$

$$= f(y)$$

$$\int_N \omega_1 \dots \omega_n = \int_{\mathcal{E}[1]\times X} dx\, d\psi\, d\pi\, dp\; \exp\left(i p_a F^a + i\pi_a \frac{\partial F^a}{\partial x^j}\psi^j\right) \cdot \omega_1(x,\psi)\cdot \dots \cdot \omega_n(x,\psi)$$

I am going to use this formula as a definition of the r.h.s.

In future X would be infinite dim, $\mathcal{E}[1]$ would also be ∞-dim, while N will still be <u>finite dim.</u>

Important development of the idea:
How to get

$$\int \mathcal{D}X\, \mathcal{D}\Psi\, \mathcal{D}p\, \mathcal{D}\Pi \, \exp\left(i p_a F^a + i \Pi_a \frac{\partial F}{\partial x^j} \Psi^j\right) \omega_1 \ldots \omega_n \cdot p_a \quad ?$$

Trick: consider family of functions

$$F_\ell^a = F^a + \ell^a$$

Then ? with no Π_b is just

$$\frac{\partial}{\partial \ell^a} \int \mathcal{D}X\, \mathcal{D}\Psi\, \mathcal{D}p\, \mathcal{D}\Pi \, \exp\left(i p_a F_\ell^a + i \Pi_a \frac{\partial F}{\partial x^j} \Psi^j\right) \omega_1 \ldots \omega_n = \frac{\partial}{\partial \ell^a} \int_{N_\ell} \omega_1 \ldots \omega$$

N_ℓ - set of zeroes of functions F_ℓ^a ; we will call N_ℓ - deformed instantons

How to insert Π? Recall the scientific meaning of $pF + \Pi F' \Psi$

We have a DeRham dif $Q = \Psi \frac{\partial}{\partial X} + p \frac{\partial}{\partial \Pi}$ on $\varepsilon[1] X$, and

$$Q(i \Pi_a F^a) = p_a F^a + \Pi_a \frac{\partial F}{\partial x^j} \Psi^j$$

Promote Q to

$$\tilde{Q} = \Psi \frac{\partial}{\partial X} + p \frac{\partial}{\partial \Pi} + \lambda \frac{\partial}{\partial \ell}$$

introducing odd coord. on family of instantons N_ℓ

$$\tilde{Q}(i \Pi_a F_\ell^a) = p_a F_\ell^a + \Pi_a \frac{\partial F_\ell}{\partial x^j} \Psi^j + \Pi_a \frac{\partial F_\ell}{\partial \ell_\alpha} \lambda^\alpha$$

$F_\ell^a = F^a + \ell^a$, then $\parallel$

$$p_a F^a + \Pi_a \frac{\partial F}{\partial x^j} \Psi^j + p_a \ell^a + \Pi_a \lambda^a$$

$$\int \exp(\qquad)\, \omega_1 \dots \omega_n \pi_a =$$

$$= \frac{\partial}{\partial \lambda^a} \int_{N_{\ell,\lambda}} \omega_1 \dots \omega_n .$$

Again, I turned would be ∞-dimensional integral into derivatie of the f. dim. integral ove $N_{\ell,\lambda}$.

Consider manifold Y and manifold X as a space of maps

$$[0, T] \longrightarrow Y$$

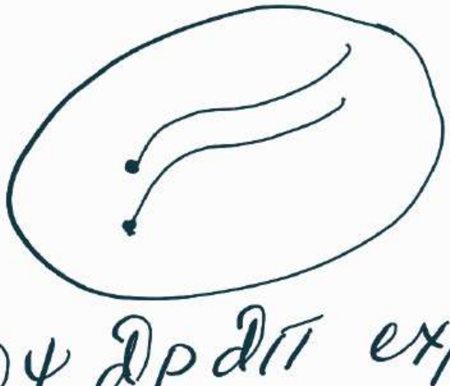
X is infinite dimensional

Equations $\dfrac{dy^i}{dt} - v^i(y(t)) = 0$

F_a

Space of solutions — trajectories of vector field — space N

Note, that $N \cong Y$



$p^* F_a \to \int dt$

$$\int \mathcal{D}y\, \mathcal{D}\psi\, \mathcal{D}p\, \mathcal{D}\pi \exp i\Big[\int p_j(t) \frac{dy^j}{dt} dt - p_j(t)\, v^j(y(t))\, dt + \pi_j \frac{d\psi^j}{dt} dt - \pi_j \frac{\partial v^j}{\partial y^k} \psi^k dt\Big]$$

$\omega_1(y(t_1), \psi(t_1)) \dots \omega_n(y(t_n), \psi(t_n)) =$

simple case, no π or p insertions

It may be computed as follows

Let $\gamma(t)$ be the trajectory

starting at y_0: $\gamma^j(0) = y_0^j$.

$$= \int_{Y} dy_0 \, d\psi_0 \; \omega_1\left(\gamma^j(t_1), \frac{\partial \gamma^j}{\partial y_0^k} \psi_0^k\right) \cdot \dots \cdot \omega_n =$$

$$= \int_{Y} ev_{t_1}^* \omega_1 \cdot \dots \cdot ev_{t_n}^* \omega_n$$

Note, that sum of degrees of ω should be equal to $\dim Y$.

<u>Example:</u>

$Y = \mathbb{R}$ $\quad n = 2$

$\omega_1 = \delta(y - y_1)\, dy$

$\omega_2 = f(y)$, Let $V = V_0$

$$\gamma(t; y_0) = y_0 + t V_0;$$

$$\int dy_0 \, d\psi_0 \; \delta(y_0 + t_1 V_0 - y_1)\, \psi_0 \, f(y_0 + t_2 V_0) =$$

$y_0 = y_1 - t_1 V_0$

$$= f(y_1 + (t_2 - t_1) V_0) \dots$$

Result depends on t_2 and t_1 if $V_0 \neq 0$

Study p - observables

Study a family

F_ℓ, well, I will study

$$F_\ell = \frac{dy^j}{dt} - v^j(y) - \ell\, u^j(y)\, \delta(t-\tau)$$

It would correspond to

an observable

$$\mathcal{O}_u(\tau) = p_j\, u^j(y) + \pi_j \frac{\partial u^j}{\partial y^k} \psi^k$$

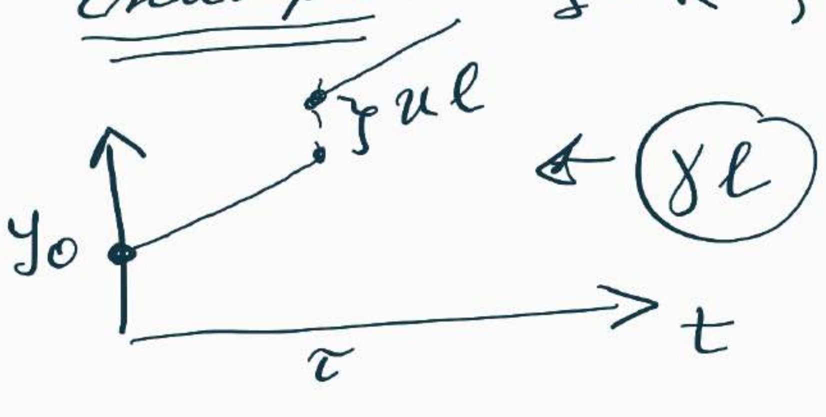
$$\langle \mathcal{O}_u(\tau)\, \omega_1(t_1) \ldots \omega_n(t_n) \rangle \overset{\text{def}}{\equiv}$$

$$\equiv \frac{\partial}{\partial \ell}\Big|_{\ell=0} \int_{N_\ell} ev^*_{t_1}\omega_1 \cdot \ldots \cdot ev^*_{t_n}\omega_n$$

$$\gamma_\ell : \frac{dy^j}{dt} - v^j(y) - \ell\, \delta(t-\tau)\, u^j = 0$$

Example: $Y = \mathbb{R}$; $v^j(y) = v_0$

$u^j(y) = u \Rightarrow$ also constant



The same computation,
for simplicity take $t_1 = 0$;
trajectory will start at y_1. we study:

$$< \delta(y - y_1)(0) \; f(t_2) \; O_u(\tilde{\tau}) > =$$

$$= \frac{\partial}{\partial \ell} \; f(\gamma_\ell(t_2)) =$$

Two cases. 1) $t_2 < \tilde{\tau}$
then the result is just

$$\frac{\partial}{\partial \ell} f(y_1 + t_2 v_0) = 0$$

2) $t_2 > \tilde{\tau}$

$$\frac{\partial}{\partial \ell} f(y_1 + t_2 v_0 + u \cdot \ell) =$$

$$= \frac{\partial f}{\partial y}(y_1 + t_2 v_0) \cdot u \quad \left(\begin{smallmatrix}**\\**\end{smallmatrix}\right)$$

So, we have a jump
The value of this jump dep. on f:
Let us take, for simp. $f(y) = y$

$\Rightarrow u$ $\langle \delta \, y(t_2) \, O_u(\tilde{\tau}) \rangle$ has
a jump, value of this jump is u

the value of a jump is indep of v_0 in this case, but in
general case: jump in $\langle \delta \, f(t_2) \, O_u \rangle$

$$= \langle \delta \, \frac{\partial f}{\partial y}(t_2) \rangle \cdot u$$

We may compare result with functorial definition of Q. Mechanics:

Dictionary:

1) Space of states $\leftrightarrow \Omega^{\bullet} Y$.

2) Hamiltonian $\leftrightarrow \mathcal{L}_{v_0}$ - Lie derivative

3) Observables $\mathcal{O}_{\omega}$ $\leftrightarrow$ operator of multiplication by $\omega\wedge$

4) Observable $\mathcal{O}_{v}$ $\leftrightarrow$ Lie derivative $\mathcal{L}_{v}$

5) Correlators:

$$\langle \Psi_{\infty} | \dots \mathcal{O}_2 \, e^{(t_2-t_1)\mathcal{L}_v} \mathcal{O}_1 \, e^{t_1\mathcal{L}_v} | \Psi_0 \rangle$$

in functorial approach.

Statement: Instantonic approach reproduces functorial approach on curved Y.

Illustration: the value of the jump in functorial approach is a commutator $[\mathcal{O}_1, \mathcal{O}_2]$ - that we observed

e^{tL_v} is the flow along the vector field.