Hi, new style of talks

More on funct. integral in simplest systems.

1) Quantization using F. I. for free particle
2) Oscillator
3) Ferm. oscillator (Berezin integral)

$$\int \mathcal{D}x \exp \frac{i}{\hbar}\int_0^T \frac{m\dot{x}^2}{2} dt = \underline{I}(x_0, x_T)$$

$x(0)=x_0$
$x(T)=x_T$

we interpret I as an integral kernel of the operator $e^{\frac{itH}{\hbar}}$; $\underline{I}(x_0, x_T) = \frac{1}{\sqrt{T}} e^{\frac{im|x_T - x_0|^2}{2T}}$

we showed that functoriality holds

$$\int dx_{T_1} \underline{I}(x_0, x_{T_1}) \cdot \underline{I}(x_{T_1}, x_{T_1+T_2}) = \underline{I}(x_0, x_{T_1+T_2});$$

moreover, we checked that $\left(i\hbar \frac{\partial}{\partial T} - \left(i\hbar \frac{\partial}{\partial x_0}\right)^2\right) \cdot \underline{I} = 0$

Sch. equation

Local observables in F. picture

$$\int \mathcal{D}x \, \mathcal{O}(x(t_1)) \exp \frac{i}{\hbar} \int \frac{m\dot{x}^2}{2} dt = \langle \mathcal{O}_{t_1} \rangle$$

this should correspond to

$0 \quad t_1 \quad T \longrightarrow \hat{\mathcal{O}}.$

Let us find it for different $\mathcal{O}(X)$: a) $\mathcal{O}(X) = X$

$$\int\limits_{\substack{x(0)=x_0 \\ x(T)=x_T}} \mathcal{D}x \quad x(t_1) \; \exp \frac{i}{\hbar}\int \frac{m^2}{2}\dot{x}^2$$

$$x(t) = x_{el} + \xi$$

$$\xi(0) = 0 \qquad \xi(T) = 0$$

$$x_{el}(t_1)\int \mathcal{D}\xi \, \exp \frac{i}{\hbar}\int \frac{m\,\dot{x}_{el}^2}{2} + \frac{i}{\hbar}\int \frac{m\,\dot{\xi}^2}{2} +$$

$$+ \underline{\int \mathcal{D}\xi \cdot \xi \; \exp \frac{i}{\hbar}\int \frac{m\,\dot{x}_{el}^2}{2} + \frac{i}{\hbar}\int \frac{m\,\dot{\xi}^2}{2}} = 0$$

$$x_{cl}(t_1)\left[\frac{1}{\sqrt{T/m}} \exp \frac{i}{\hbar} \frac{m(x_T-x_0)^2}{2T}\right]$$

$\vdash\!\dashv$ → small interval $T \to 0$

tend to $\delta(x_T - x_0)$

$$\int \mathcal{D}x\, x(t) \exp \frac{i}{\hbar} S'(x) \to x_T\, \delta(x_T - x_0)$$

Operator, corresp to $x(t)$ is just a multiplication by x

$$\int \mathcal{D}x\, m\dot{x}(t_1) \exp \frac{i}{\hbar} S'(x) \qquad \boxed{\dot{x} = \frac{dx}{dt}}$$

$$= m \frac{d}{dt} x_{cl} \cdot \frac{1}{\sqrt{T}} e^{\frac{im}{2\hbar}(x_T - x_0)^2} =$$

$$x_{cl} = t \cdot \frac{x_T - x_0}{T}$$

$$\frac{dx_{cl}}{dt} = \frac{x_T - x_0}{T}$$

$$m\frac{(x_T - x_0)}{T} \cdot \frac{1}{\sqrt{T}} e^{\frac{im}{2\hbar}(x_T - x_0)^2} - ?$$

Computation: $i\hbar \frac{\partial}{\partial x_T} \frac{1}{\sqrt{T}} \exp \frac{im}{2\hbar}(x_T - x_0)^2 =$

$$= \frac{1}{\sqrt{T}} \cdot \frac{m}{\cancel{2}} \cancel{2}(x_T - x_0) \cdot \exp \frac{im}{2\hbar}(x_T - x_0)^2$$

Feynman quantization by F.I.

$$m\dot{x} \to i\hbar \frac{\partial}{\partial x_T} \,!$$

Try to generalize to QFT:

Feynm. idea:

$$\langle O_1(x_1) \dots O_n(x_n) \rangle =$$

x_i are points on the space-time

$$= \int \mathcal{D}\varphi \prod_{i=1}^{n} F_i(\varphi, \partial_\mu \varphi, \partial^2_{\mu\nu}\varphi \dots)$$

$$\exp \frac{i}{\hbar} S$$ (Problems come out)

$$S = \int \mathcal{L}(\varphi, \partial\varphi, \partial^2\varphi, \dots)$$

when we do it naively this diverges!!!

Origin of divergence

Let $\mathcal{L}$ be quadratic, such like $d\varphi * d\bar{\varphi}$

No problems to define

$$\int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \; \varphi(x_1)\dots\varphi(x_n) \; \bar{\varphi}(y_1)\dots\bar{\varphi}(y_n) \exp \frac{i}{\hbar}\int d\varphi * d\bar{\varphi} =$$

Wick formula

$$\sum_{\sigma \in S_n} G(x_1, y_{\sigma(1)}) \dots G(x_n, y_{\sigma(n)}) \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \exp \frac{i}{\hbar} \int d\varphi * d\bar{\varphi}$$

S_n ← sym. group

where $G(x,y)$ are Green's functions

$$\Delta G(x,y) = \delta(x-y)$$

Problem appears when

$$\mathcal{O}(\varphi) = \underline{\underline{\varphi\bar{\varphi}}} \; : \; y \to x$$

$G(x,x)$ diverges!!

Origin of UV problems in Feynman QFT.

IR problem: zero mode

$$d * d\, G(x,y) = \delta(x-y)$$ has no solutions

$$\int_X \text{l.h.s} = 0 \qquad \int_X \text{r.h.s} = 1.$$

$$d * d\, G(x,y) = \delta(x-y) - e(x)\cdot e(y)$$

$e(x)$ - shape of the zero mode

e may be understood as follows:

$$I_m = \int \mathcal{D}\varphi \, \exp \int \left(d\varphi * d\bar{\varphi} + m^2 |\varphi|^2 \cdot e_1(x) \dots e(x) \right)$$

$\sqrt{\det g}$

I_m has no zero mode; I_m, then take $m \to 0$

result would diverge like $\frac{1}{m}$

$$I^{\text{renorm}} = \lim_{m \to 0} m \cdot I_m \to \text{IR renorm.}$$

Interestingly,
shape of the massive "regulator" $|\varphi|^2 m\sqrt{g}$ survives in the answer.
(IR anomaly)

By the way, if X has a boundary, and we just put $\varphi|_{\partial X}=0$ we do not need to do IR renormalization like we did before $X(T)=X(0)=0$

Problems of F.I. definition of QFT.

a) Naively defined observables like $F(\varphi,\bar{\varphi})$ diverge under the integral

b) $\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}\int_X d\varphi * d\bar{\varphi} + F(\varphi,\bar{\varphi})} \stackrel{?}{=}$

$$\int_{x_1\in X}\cdots\int_{x_n\in X}\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}\int d\varphi * d\varphi} F(\varphi,\bar{\varphi})(x_1)\dots F(\varphi,\bar{\varphi})(x_n)$$

when $x_i \to x_j$ k

this diverges due to $G(x_i,x_j)$

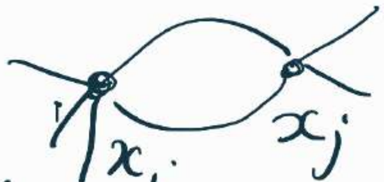
physicists write it like this

Green functions (k of them)

if $k=1$

singularity lik $G(x_i,x_j)$ is integr.

if $k = 2$



then singularity $G(x_i, x_j)^2$ is nonintegrable! (physicists call it loop divergences)

Example: $\dim X = 4$ $\quad G(x,y) = \frac{1}{|x-y|^2}$

$G(x,y) \cdot G(x,y) = \frac{1}{|x-y|^4}$ $\quad \int d^4x \frac{1}{|x-y|^4}$

logarithmically diverges!!!

It actually happens in physics (Higgs boson theory)

basically $\mathcal{L} = d\varphi * d\bar{\varphi} + |\varphi|^4$

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way out → old way – renormalize → some ugly procedure that is an additional prescription added to the "definition" of F.I.

$$\int \mathcal{D}\varphi \, e^{\frac{i}{\hbar}S(\varphi)} + \text{extra prescription.}$$

New way → go to functorial view on QFT where problem a) just does not happen: we will say that $|\varphi|^2$ is not a local observable → discuss it later.

problem b) $\langle \mathcal{O}(x_i) \, \mathcal{O}(x_j) \rangle$ diverges when $x_i \to x_j$ and cannot be integrated

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when we deform QFT

$$\langle \ \rangle_0 \to \langle \ \rangle_0 + \epsilon \langle \int_X \mathcal{O} \rangle_0$$

we should also deform the local observables

$$\mathcal{O}_0 \to \mathcal{O}_0 + \epsilon \, \delta \mathcal{O}_0$$

Then the second order deformation would be just a first order deformation in the deformed theory! So no conceptual problems.

we will see how it works later.

Another example where F.I. works – oscillator

$$\mathcal{L} = \frac{m\dot{X}^2}{2} - K\frac{X^2}{2} \to \text{after some rescaling}$$

$$\mathcal{L} = \frac{1}{2}(\dot{X}^2 - \omega^2 X^2)$$

Computation by F.I. of maps $S^1 \to R$

$$\int_{X(0)=X(T)} \mathcal{D}X \ \exp \frac{i}{\hbar}\int(\dot{X}^2 - \omega^2 X^2)\,dt$$

ω – parameter

Gaussian integral that is expected to corresp. to $\mathrm{tr}\, e^{\frac{iT}{\hbar}H}$

$$X = \sum_n X_n \underbrace{e^{2\pi i n \frac{t}{T}}}_{\exp(n)} \qquad X_n = \overline{X}_{-n}$$

$$\boxed{S' = T \sum_{n>0}\left(\frac{n^2}{T^2} - \omega^2\right) X_n \overline{X}_n} \quad (*)$$

for special ω – zero modes

$$\dot{X}(t) = \sum_n 2\pi i \frac{n}{T} X_n \exp(n)$$

$$\int dt\, \dot{X}(t)^2 = \int dt \Big[\sum_{n,n'} 2\pi i \frac{n}{T}\, 2\pi i\, (t n')\, X_n X_{n'} \exp(n) \exp(n')\Big]$$

$$-\sum_{n,n'} 4\pi^2 n n' \int \exp(n) \exp(n')\, dt \cdot X_n X_{n'}$$

$$\int \exp(n) \exp(n')\, dt = \delta_{n+n',0} \cdot T$$

$$= \sum_{n\neq 0} 4\pi^2 n^2 X_n X_{-n} = \sum_{n\neq 0} 4\pi n^2 X_n \overline{X}_n$$

case $n=0$ - separate case

$$X_0^2 \omega^2$$

$$\int \left(\prod_{n>0} dX_n d\overline{X}_n\right) dX_0 \exp \omega^2 T X_0^2 \cdot \exp\left[T\left(4\pi^2 \frac{n^2}{T^2} - \omega^2\right) X_n \overline{X}_n\right] = \overline{1}$$

Formally, the answer is

$$\frac{1}{\omega \sqrt{T}}^{*} \prod_{n>0}^{\infty} \frac{1}{T\left(4\pi^2 \frac{n^2}{T^2} - \omega^2\right)} = -$$

$\omega \to 0$ Ікаким in $d=1$

Studied it when $\omega = 0$

Here - general case $z = \omega \cdot T$

$$\prod_{n>0}^{\infty} \frac{\overline{1}}{(4\pi n^2 - z^2)} \quad \frac{1}{\omega\sqrt{T}} \stackrel{?}{=}$$

we would compare it with the standard formula for the

$\sin z \to$ to be continued tomorrow. (denominator is like $\sin z$)...