Linear Generalized $\mathbb{C}$ Structures

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3. Defining linear g. $\mathbb{C}$ structures as...
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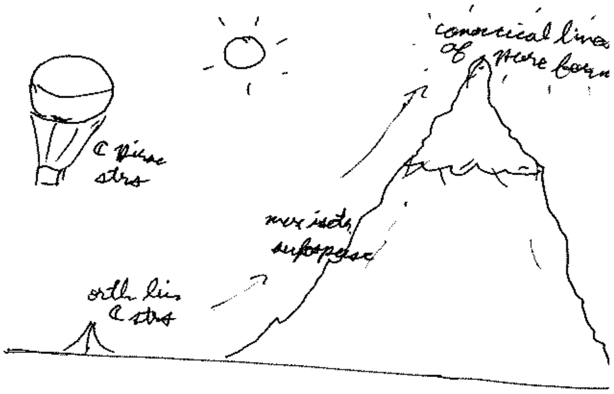
   ii. maximal isotropic subspaces
   iii. $\mathbb{C}$ Dirac strs
   (iv. canonical lines of pure forms)
4. (Preview:) The linear B-field transform



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1. Introduction

* Hitchin: The idea of generalized geometry is to replace...

$$TM \longrightarrow TM \oplus T^*M$$
"extended tangent bundle"

$$[\,,\,] \longrightarrow [\![\,,\,]\!]$$
Courant bracket

* This has applications in physics (string theory, supersymmetry, T-duality)

* g. $\mathbb{C}$ geometry encompases <u>both</u> complex $\mathbb{C}$ geometry & sym. geometry as special cases.

$\mathbb{C}$ str ——— g. $\mathbb{C}$ str ——— sym str

<u>def.</u> A generalized $\mathbb{C}$ structure on M is

today → ~~fiberwise condition~~ structure: * An smooth assignment $\mathcal{J}$ of orthogonal linear $\mathbb{C}$ strs to each fiber of $TM \oplus T^*M \longrightarrow M$

such that

global condition: * the $+i$-eigenbundle of $\mathcal{J}$ is Courant integrable.

2. The Double

V - vector space

<u>def</u>. The <u>double</u> of V comprises

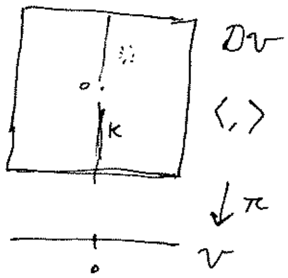
i. a $2n$-dim v.s. DV,

ii. a nondegenerate symmetric pairing $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$,

iii. a projection $\pi : DV \longrightarrow V$,

such that

* $\underbrace{\ker \pi}_{K} \subseteq DV$ is <u>isotropic</u>, i.e $\langle , \rangle|_{\ker \pi} = 0$.



wts. $K \subseteq DV$ is <u>maximal</u> isotropic.

<u>def</u>. The <u>orthogonal</u> of a subspace $A \subseteq (E, \langle , \rangle)$ is
$$A^\perp := \{ e \in E \mid \langle A, e \rangle = 0 \}.$$

<u>fact</u>. $A \subseteq (E, \langle , \rangle)$ is isotropic $\iff A \subseteq A^\perp$.

<u>lem</u>. i. $\dim A + \dim A^\perp = \dim E$

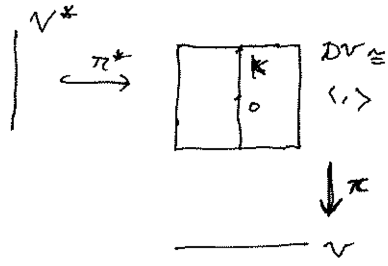
ii. $A \subseteq E$ isotropic $\implies \dim A \leq \frac{1}{2} \dim E$.

<u>prf</u>. i. Define
$$\varphi : E \longrightarrow A^*$$
$$e \longmapsto \langle e, \ \rangle|_A .$$
$$\dim \underbrace{\ker \varphi}_{A^\perp} + \dim \underbrace{\operatorname{im} \varphi}_{A^*} = \dim E.$$

ii. $A \subseteq A^\perp \implies \dim A \leq \dim A^\perp \overset{}{\underset{i.}{\implies}} \dim A + \dim A^* \leq \dim E$. □

~~cor~~ <u>prop</u>. $K \subseteq DV$ is max. isotropic.

Equivalently, $K = K^\perp$.



$DV \cong V \oplus V^*$
$\langle , \rangle$

$$(DV)^* \cong DV$$
$$\uparrow \pi^*$$
$$V^*$$

prop. $\mathrm{im}\, \pi^* = \ker \pi$

prf. suffices to show $\subseteq$. $e \in DV$

$$e \in \mathrm{im}\, \pi^* \Rightarrow \langle e, \ \rangle = \pi^* \underset{\alpha}{\overset{V^*}{\in}}$$
$$\Rightarrow \langle e, k \rangle = (\pi^* \alpha)(k) = \alpha(\underbrace{\pi k}_{0}) = 0$$
$$\Rightarrow e \in K^\perp = K$$

$\square$

$\therefore$ s.e.s.

$$0 \longrightarrow V^* \xrightarrow{\pi^*} DV \xrightarrow{\pi} V \longrightarrow 0$$

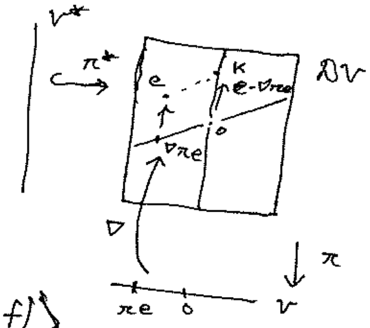
ex $DV = V \oplus V^*$, $\langle , \rangle_{can}$

$$\langle (v, \alpha), (v', \alpha') \rangle_{can} := \alpha'(v) + \alpha(v').$$

~~prop. If ∇ V a (right) spl~~

prop. an isotropic (right) splitting $\nabla : V \longrightarrow DV$ yields an iso.

$$\varphi : (DV, \langle , \rangle) \xrightarrow{\sim} (V \oplus V^*, \langle , \rangle_{can})$$
$$e \longmapsto (\pi e, e - \nabla \pi e).$$



prf. $e, f \in DV$,

$$\langle \varphi e, \varphi f \rangle_{can} = \langle (\pi e, e - \nabla \pi e), (\pi f, f - \nabla \pi f) \rangle_{can}$$
$$= \langle f - \nabla \pi f, e \rangle + \langle e - \nabla \pi e, f \rangle$$
$$= \langle f, e \rangle + \langle e, f \rangle \underbrace{- \langle \nabla \pi f, e \rangle - \langle \nabla \pi e, f \rangle}_{\overset{=}{(*)} - \langle e, f \rangle}$$

$$(*) \quad 0 = \langle \nabla \pi e - e, \nabla \pi f - f \rangle$$
$$= \underbrace{\langle \nabla \pi e, \nabla \pi f \rangle}_{0} - \langle e, \nabla \pi f \rangle - \langle \nabla \pi e, f \rangle + \langle e, f \rangle$$

$\square$

3. defining Generalized linear $\mathbb{C}$ strs as...

i. orthogonal linear $\mathbb{C}$ strs

<u>def.</u> a linear <u>generalized $\mathbb{C}$ str</u> on V is

· a linear $\mathbb{C}$ str $\mathcal{J}$ on $\mathbb{D}V$

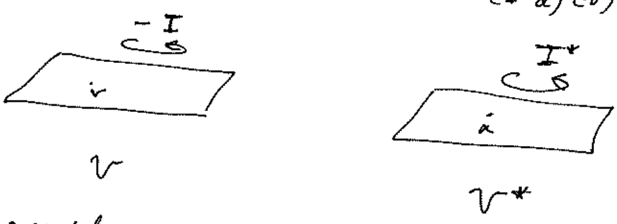
such that

* $\mathcal{J}$ preserves $\langle , \rangle$, i.e. $\forall e,f \in \mathbb{D}V$: $\langle \mathcal{J}e, \mathcal{J}f \rangle = \langle e,f \rangle$.

<u>def.</u> a <u>linear $\mathbb{C}$ str</u> on E is an endomorphism $J \in \operatorname{End} E$ s.t. $J^2 = -1_E$

<u>ex.</u> $I \in \operatorname{End} V$ — linear $\mathbb{C}$ str on V.

$$\mathcal{J}_I := \begin{pmatrix} -I & \\ & I^* \end{pmatrix}, \qquad \begin{aligned} I^*\alpha &= \alpha(I\,\cdot) \\ \forall v \in V:\ (I^*\alpha)(v) &= \alpha(Iv) \end{aligned}$$



~~check~~ verify:

$$\mathcal{J}_I^{\,2} = \begin{pmatrix} -I & \\ & I^* \end{pmatrix}^2 = \begin{pmatrix} +I^2 & \\ & (I^*)^2 \end{pmatrix} = \begin{pmatrix} -1_V & \\ & -1_{V^*} \end{pmatrix} = -1_{\mathbb{D}V} \quad \checkmark$$

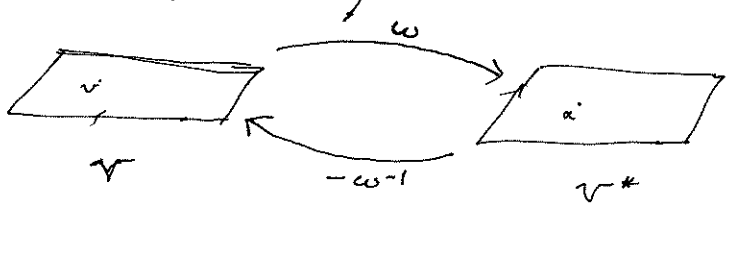
$$\begin{aligned} \langle \mathcal{J}_I(v+\alpha), \mathcal{J}_I(v'+\alpha') \rangle &= \langle -Iv + I^*\alpha, -Iv' + I^*\alpha' \rangle \\ &= (I^*\alpha')(-Iv) + (I^*\alpha)(-Iv') \\ &= \alpha'(v) - \alpha(v') \\ &= \langle v+\alpha, v'+\alpha' \rangle. \quad \checkmark \end{aligned}$$

<u>ex.</u> ~~ω = linear~~ $\omega \in \Lambda^2 V^*$ — linear sym. str on V.

i.e. $\omega: V \xrightarrow{\sim} V^*$

$v \longmapsto \iota_v \omega = \omega(v, \cdot)$,

$$\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$$



verify:

$$\mathcal{J}_\omega^2(v+\alpha) = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}^2 \begin{pmatrix} v \\ \alpha \end{pmatrix} = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix} \begin{pmatrix} -\omega^{-1}\alpha \\ \omega v \end{pmatrix}$$

$$= \begin{pmatrix} -\omega^{-1}(\omega v) \\ -\omega(\omega^{-1}\alpha) \end{pmatrix} = -\begin{pmatrix} v \\ \alpha \end{pmatrix} = -(v+\alpha) \checkmark$$

$$\langle \mathcal{J}_\omega(v+\alpha), \mathcal{J}_\omega(v'+\alpha') \rangle = \langle \omega v - \omega^{-1}\alpha, \omega v' - \omega^{-1}\alpha' \rangle$$

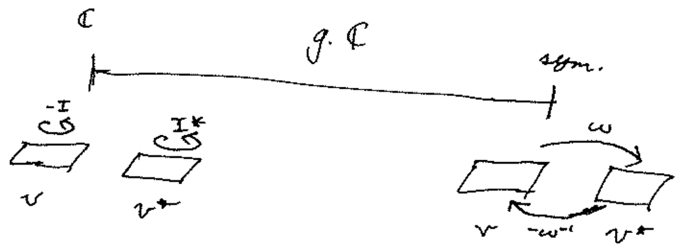
$$= \omega v(-\omega^{-1}\alpha') + \omega v'(-\omega^{-1}\alpha)$$

$$= -\iota_{\omega^{-1}\alpha'}\iota_v\omega - \iota_{\omega^{-1}\alpha}\iota_{v'}\omega$$

$$= \iota_v\iota_{\omega^{-1}\alpha'}\omega + \iota_{v'}\iota_{\omega^{-1}\alpha}\omega$$

$$= \iota_v\alpha' + \iota_{v'}\alpha$$

$$= \langle v+\alpha, v'+\alpha' \rangle \checkmark \ \square$$



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ii. maximal isotropic subspaces

fact. a ~~linear ℂ~~ linear ℂ str $J \in \operatorname{End} E$ ~~on E splits E~~ yields a splitting

$$E^{\mathbb{C}} = \underset{+i \text{ eigenspace}}{E^{1,0}} \oplus \underset{-i}{E^{0,1}}.$$

prf. $\{e_k\}_k \subseteq E$ basis $\Longrightarrow$ $\{\underbrace{e_k - iJe_k}_{\in E^{1,0}\,(*)}, \underbrace{e_k + iJe_k}_{E^{0,1}}\}_k \subseteq E^{\mathbb{C}}$ basis

(*) $J(e_k - iJe_k) = Je_k - i\underbrace{J^2}_{-1}e_k = Je_k + ie_k = i(e_k - iJe_k)$ $\square$

Thus $(\mathbb{D}V)^{\mathbb{C}} = \underset{+i}{L} \oplus \underset{-i}{\bar{L}}$ $\mathcal{J}$-eigenspace

$\langle\ ,\ \rangle$ on $\mathbb{D}V$ extends to $(\mathbb{D}V)^{\mathbb{C}}$ by $\mathbb{C}$-linearity.

WARNING: $\langle , \rangle$ on $(\mathbb{D}V)^{\mathbb{C}}$ is not a Hermitian str.

prop. $L, \bar{L} \subsetneq (\mathbb{D}V)^{\mathbb{C}}$ are max. isotropic subspaces.

$(\mathbb{D}V)^{\mathbb{C}} = L \oplus \bar{L}$ is a max isotropic splitting.

pf. $v, w \in L$.

$$\langle v, w\rangle = \langle \mathcal{J}v, \mathcal{J}w\rangle = \langle iv, iw\rangle = -\langle v, w\rangle$$

$$\Rightarrow \langle , \rangle|_L = 0.$$

$$\underbrace{\dim L}_{=\dim_{\mathbb{R}} \mathbb{D}V} = \tfrac{1}{2}\dim (\mathbb{D}V)^{\mathbb{C}} \Rightarrow L \text{ is max. isotropic. } \square$$

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prop. There is a 1-1 correspondence,

$$\{\text{orth. } \mathbb{C} \text{ strs on } \mathbb{D}V\} \xleftrightarrow{1\text{-}1} \{\text{max. isotropic splittings } \mathbb{D}V^{\mathbb{C}} = L \oplus \bar{L}\}$$

$$\mathcal{J} \longmapsto \mathbb{D}V^{\mathbb{C}} = \underset{+i}{L} \oplus \underset{-i}{\bar{L}} \quad \text{eigenspaces}$$

$$\mathcal{J}_L = \begin{pmatrix} i & \\ & -i \end{pmatrix}\Big|_{\mathbb{D}V} \longleftarrow \mathbb{D}V^{\mathbb{C}} = L \oplus \bar{L}$$

pf. Claim $\mathcal{J}_L$ ~~preserves~~ $\circlearrowright \mathbb{D}V$.

note: $v \in \mathbb{D}V \Leftrightarrow v = \bar{v} \Leftrightarrow v = w + \bar{w}$ for some $\overset{\pi_L v}{w} \in L$.

$$\boxed{\left.\begin{array}{l} v = \underset{\in L}{v^+} + \underset{\in \bar{L}}{v^-} \\ \bar{v} = \underset{\in \bar{L}}{\overline{v^+}} + \underset{\in L}{\overline{v^-}} \end{array}\right\} \Rightarrow v^- = \overline{v^+}}$$

$$\mathcal{J}_L v = iw - i\bar{w} = iw + \overline{iw} \in V \quad \checkmark \quad \square \text{ of claim}$$

verify:

$$\begin{pmatrix} i & \\ & -i \end{pmatrix}^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \quad \checkmark$$

$$\langle \mathcal{J}_L(e^+ + e^-), \mathcal{J}_L(f^+ + f^-)\rangle = \langle ie^+ - ie^-, if^+ - if^-\rangle$$

$$= -\langle e^+ - e^-, f^+ - f^-\rangle = \langle e^+, f^-\rangle + \langle e^-, f^+\rangle = \langle e^+ + e^-, f^+ + f^-\rangle \ \checkmark$$

$\square$

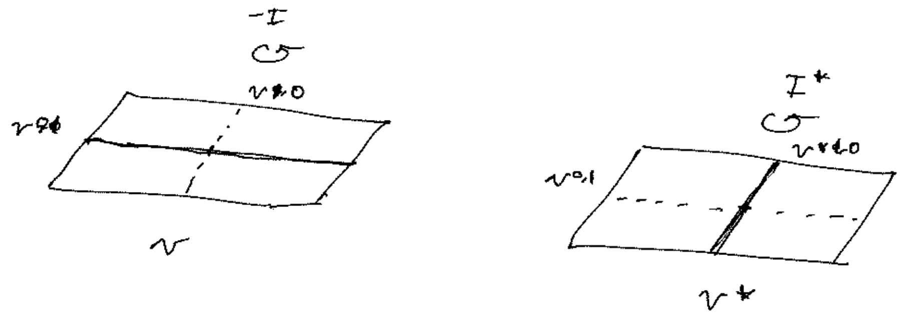
ex. $I \in \mathrm{End}\,V$. $\mathbb{C}$ str on V.

$$V = \underset{+i}{V^{1,0}} \oplus \underset{-i}{V^{0,1}}$$

$$V^* = V^{*1,0} \oplus V^{*0,1} \qquad , \qquad \mathcal{J}_I = \begin{pmatrix} -I & \\ & I^* \end{pmatrix}$$

$$L = V^{0,1} \oplus V^{*1,0}$$

$$\bar{L} = V^{1,0} \oplus V^{*0,1}$$



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ex. $\omega \in \Lambda^2 V^*$ ~~sym~~ str.

$$\mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}$$

~~$L = \{ e - i\mathcal{J} e$~~

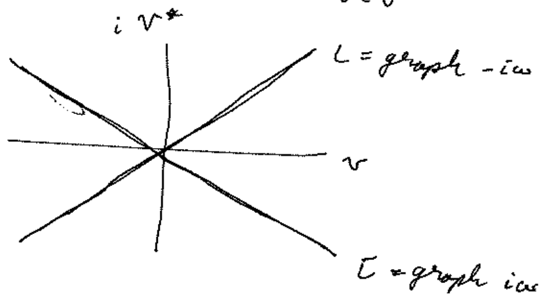
$$L = \{ (v+\alpha) - i\mathcal{J}(v+\alpha) \mid \cancel{v \in V, \alpha \in} \; v+\alpha \in \mathbb{D}V \}$$

$$= \{ (v - i\omega v) + \underbrace{(\alpha + i\omega^{-1}\alpha)} \mid v \in V, \underbrace{\alpha \in V^*} \}$$

redundant ($v = \omega^{-1}\alpha$)

$$= \{ v - i\omega v \mid v \in V \}.$$

$$\bar{L} = \{ v + i\omega v \}_{v \in V}$$



$\mathbb{D}V = L \oplus \bar{L}$ is

isotropic: $\langle v - i\omega(v), v' - i\omega(v') \rangle = -i\omega(v)v' - i\omega(v')v$

$$= -i\left[\iota_{v'}\iota_v\omega + \iota_v\iota_{v'}\omega \right] = 0.$$

iii. $\mathbb{C}$ Dirac strs

def. A linear Dirac str on V is a max. isotropic subspace ~~of~~ $A \subseteq DV$.

Thus a g. $\mathbb{C}$ str $L \subseteq (DV)^{\mathbb{C}}$ is a complex Dirac str. with a $\mathbb{C}$ compatibility cond $L \cap \bar{L} = 0$.

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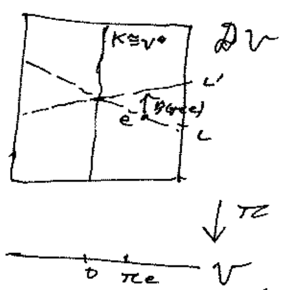
4. Preview: The B-field transform

So far: examples & characterizations of g. C strs.

A 2-form $B \in \Lambda^2 V^*$ yields a B-field transform

$$\begin{aligned} DV &\to DV \\ e &\mapsto e - B(\pi e) \end{aligned}$$



V. $B \in \Lambda^2 V^*$

* B carries g$\mathbb{C}$ strs to g$\mathbb{C}$ strs
* ~~E~~ Any two g$\mathbb{C}$ strs are related by some $B \in \Lambda^2 V^*$.