

Discrete Probability in models of Mathematical Physics

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A.A. colours each face of the hexagonal lattice in yellow or blue independently: in yellow with probability p ; in blue with probability $1 - p$ (left figure below).

V.V.: *Will there be an infinite connected component of yellow faces?*

A.A.: Depends on p : there is almost surely a unique infinite component when $p > 1/2$ and there is no infinite component otherwise. The model is called [Bernoulli percolation](#).

V.V.: *How do interfaces between yellow and blue components look like at $p = 1/2$?*

A.A.: Like a random fractal [SLE₆](#)! We will prove this seminal theorem of Smirnov.

V.V.: *What if the colours of faces are not independent?*

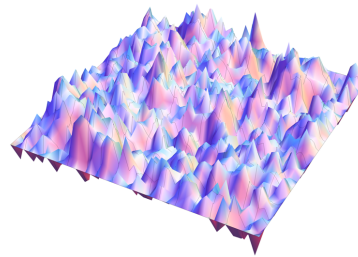
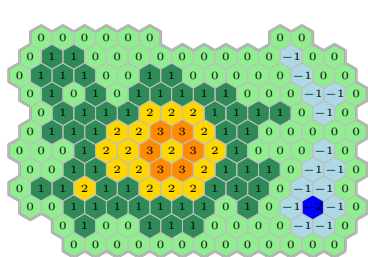
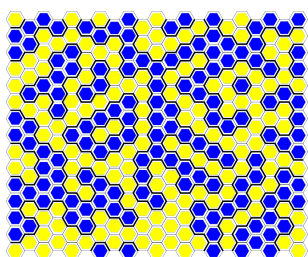
A.A.: Then you can get the [Ising model](#) for ferromagnetism: define the distribution to be proportional to $e^{-1/T}$ to the number of pairs of adjacent faces of different colour.

V.V.: *Hm... only neighbours interact? Won't we get percolation on far away faces?*

A.A.: Only for large T ! The transition occurs at [Curie temperature](#) T_c : when $T < T_c$, one of the colours dominates. Convergence to [SLE₆](#) when $T > T_c$ is a big conjecture.

V.V.: *What if we put integer numbers instead of colours?*

A.A.: View numbers as heights and get a random surface! We will discuss the uniform distribution on Lipschitz functions: they differ by 0, 1, or -1 at adjacent faces (central figure). They are conjectured to converge to the [Gaussian Free Field](#) (right figure).



Background. Though these models have simple definitions, they allow to construct deep mathematical theories. The growing interest to lattice models is confirmed with Fields Medals to [Werner](#) (2006) and [Smirnov](#) (2010). Also, [Berezinskii–Kosterlitz–Thouless](#) phase transition in the [XY model](#) was awarded Nobel Prize in Physics in 2016.

Open problems. There are a lot of new questions of various difficulty — some could become a subject for a Bachelor/Master/PhD thesis.

Prerequisites. Introduction to Probability Theory.
No previous knowledge of Physics/Mathematical Physics is assumed.