# Course on Lattice models in St. Petersburg, 2021: Exercises

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Please fill in the doodle by **12pm on Wednesday, Nov 10**. I will then choose randomly the presenters. These presenters then should (rather quickly) send me their solutions, so that I can check them. Preferably, the solutions should be written in English. If you have difficulties with this, I could help you with translation.

For the presentation: the best option is to share your solution over zoom. It's easier to understand if the presenter has their camera switched on.

The questions marked with a start $^*$ are not mandatory. It is also ok not to solve all other questions, but over all you should solve 50-60% of the mandatory questions.

Hints are marked as footnotes and appear at the end of the file.

**Graph**: the hypercubic lattice in dimension $d$ is denoted by $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$.
A box of size $n$ is denoted by $\Lambda_n$ and is defined as a subgraph of $\mathbb{L}^d$ spanned by the vertex-set $[-n, n]^d$.

**Percolation**: configuration space is $\{0,1\}^{\mathbb{E}^d}$. The $\sigma$-algebra $\mathscr{F}$ is generated by all cylinder events: these are events that depend on finitely many edges. Let $p \in [0,1]$. To each edge we assign a Bernoulli random variable $\omega_e$:

$$\mathbb{P}(\omega_e = 1) = p, \quad \text{and} \quad \mathbb{P}(\omega_e = 0) = 1 - p.$$

The percolation measure $\mathbb{P}_p$ is defined on all events in $\mathscr{F}$ as the product measure of these Bernoulli random variables. In particular, for any two disjoint finite sets of edges $E, F \subset \mathbb{E}^d$,

$$\mathbb{P}_p(\text{all edges in } E \text{ are open, all edges in } F \text{ are closed}) = p^{|E|}(1-p)^{|F|}.$$

**Exercise 1.** a) Show that $p_c(\mathbb{Z}) = 1$ (no phase transition in dimension one).
b) Show that $p_c(\mathbb{Z} \times \{0, \dots, n\}) = 1$ (no phase transition in a strip).

**Exercise 2.** Show that $p_c(\mathbb{Z}^d) \le 3/4$, for every $d \ge 2$.

For every $k \in \mathbb{N} \cup \{0\}$, fix some $p_k \ge 0$ such that $\sum_{k=0}^{\infty} p_k = 0$. Consider the following process:

- start by one individual;
- this individual has a random number of offsprings:

$$\mathbb{P}(k \text{ offsprings}) = p_k.$$

- then every individual of this first generation gives birth to a random number of offsprings with the same distribution.
- etc.

The obtained tree is called the *Galton–Watson tree* (or a branching process).

**Exercise 3.** * Consider the infinite tree of degree $d+1$. Connect the cluster of the origin to a Galton–Watson tree. Use this connection to compute the critical point.[1]

**Exercise 4.** * Recall that by $\mathrm{SAW}_n$ we denote the number of self-avoiding walks (or simple paths) starting at the origin. Denote by $c_n$ the number of walks in $\mathrm{SAW}_n$. Show that (the logarithm of) $c_n$ is sub-additive:

$$c_{n+m} \leq c_n \cdot c_m.$$

Deduce that $c_n^{1/n}$ converges as $n$ tends to infinity to $\mu_c \in (0, \infty)$. [2]

**Exercise 5.** Show that any event in $\mathscr{F}$ can be approximated by events depending on finitely many edges. That is, show that for any event $A \in \mathscr{F}$, there exists a sequence of events $B_n \in \mathscr{F}$, such that $B_n$ depends only on the edges in the box of size $n$ and

$$\mathbb{P}_p(A \Delta B_n) \xrightarrow[n\to\infty]{} 0.$$

where $A \Delta B_n := (A \setminus B_n) \cup (B_n \setminus A)$ (it is called the *symmetric difference*).[3]

**Exercise 6.** Show that the percolation measure $\mathbb{P}_p$ is ergodic.[4]

**Exercise 7.** a) When $p < p_c$, show that $\theta(p) = \psi(p) = 0$.
b) When $p > p_c$, show that $\theta(p) > 0$ and $\psi(p) = 1$.[5]

**Exercise 8.** * a) What site percolation on the square (hexagonal and triangular) lattice (in dimension two) could mean? Show that $0 < p_c < 1$.

b) Show that bond percolation on a graph corresponds to site percolation on a modified graph.

c) Consider site and bond percolations on $\mathbb{Z}^d$. Show that[6]

$$p_c(bond) \leq p_c(site) \leq 1 - (1 - p_c(bond))^d.$$

**Exercise 9.** Consider a graph $G$ for which every vertex has degree smaller than or equal to $d$. We call a finite set $S$ of vertices of $G$ a *lattice animal* if the subgraph of $G$ induced on $S$ is connected. For $x \in G$ and $n \in \mathbb{N}$ let $a(n,x)$ be the number of lattice animals with $n$ vertices that contain $x$. Show that,[7]

$$\sum_{n=0}^{\infty} [p(1-p)]^{dn} a(n,x) \leq 1.$$

Deduce that $a(n,x) \leq 4^{dn}$, for every $x \in G$ and $n \in \mathbb{N}$. Show one can replace the term $[p(1-p)]^{dn}$ in the sum by $p^n(1-p)^{dn}$.

A graph $G$ is called *transitive* if, for every two vertices $u$ and $v$, there exists an automorphism of G that is mapping $u$ to $v$. A graph $G$ is called ameanable if the following holds:

$$\inf_{H\subset G} \frac{|\partial H|}{H} = 0,$$

where the infimum is taken over all subgraphs of $G$ and $\partial H$ denotes the set of vertices in $H$ that are adjacent to vertices outside of $H$.

**Exercise 10.** a) Check that the number of trifurcation points $T$ in $\Lambda_K$ is smaller or equal that the number of vertices in $\partial\Lambda_K$. b) Check that the same proof works for any transitive amenable graph to show that the infinite cluster is unique.

Let $S$ be a set of vertices in $\mathbb{Z}^d$. Given a percolation configuration, we say that vertices $u$ and $v$ are *connected in* $S$ if there exists a path of open edges starting at $u$, ending at $v$, and such that all vertices in the path belong to $S$. We denote this event by $\{u \overset{S}{\leftrightarrow} v\}$.

**Exercise 11.** Let $S \subset \mathbb{Z}^d$ be a finite set of vertices containing the origin 0. For any $x \notin S$, show the following inequality [8]:

$$\mathbb{P}_p(0 \leftrightarrow x) \le \sum_{y \in \partial S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} y)\mathbb{P}_p(y \leftrightarrow x).$$

**Exercise 12.** Show that $\theta(p)$ is strictly increasing when $p > p_c$.[9]

**Exercise 13.** * The aim of this exercise is to show that $\theta(p)$ is continuous on $[0,1] \setminus p_c$ and right-continuous at $p_c$.

a) Define $\theta_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n$. Show that $\theta_n(p)$ is a continuous function of $p$, increasing in $p$ and decreasing in $n$, and that $\lim_{n\to\infty} \theta_n(p) = \theta(p)$, for any $p \in [0,1]$.

b) Show that a decreasing limit of continuous increasing functions is right continuous. Derive that $\theta(p)$ is right-continuous on $[0,1]$.

c) Consider a coupling of percolation measures: to all edges $e \in \mathbb{E}^d$ assign independent uniform random variables $U_e$ on $[0,1]$. Let $\mathbb{U}$ denote the joint distribution of $(U_e)_{e\in\mathbb{E}^d}$. For any $p \in [0,1]$, define a random percolation configuration $\omega^p$: for $e \in \mathbb{E}^d$, we set $\omega^p_e = 1$ if $U_e \le p$ and $\omega^p_e = 0$ if $U_e > p$.

As we have seen in the class, the distribution of $\omega^p$ is given by $\mathbb{P}_p$.

Show that, for any $p > q > p_c$

$$\theta(p) - \theta(q) = \mathbb{U}(0 \overset{\omega^p}{\longleftrightarrow} \infty, 0 \overset{\omega^q}{\not\longleftrightarrow} \infty).$$

d) Take any $p > q > p_c$. Denote by $\mathscr{C}^q$ the unique infinite cluster in $\omega^q$. Show that either $\theta(p)$ is not left-continuous at some $p$ or, for some constant $c > 0$ and for any $p' < p$,

$$\mathbb{U}(0 \overset{\omega^p}{\longleftrightarrow} \mathscr{C}^q, 0 \overset{\omega^{p'}}{\not\longleftrightarrow} \mathscr{C}^q) \ge c.$$

e) Show that the above implies that $\mathbb{P}(\exists e \in \mathbb{E}^d \, : \, U_e = p) \ge c$.

f) Get a contradiction and deduce that $\theta(p)$ is left-continuous on $[0,1] \setminus \{p_c\}$.

**Exercise 14.** a) Show that the intersection and the union of two increasing events is an increasing event.

b) Show that the set of all increasing events generate the product $\sigma$-algebra on $\{0,1\}^{E(\mathbb{Z}^d)}$.[10]

**Exercise 15.** Show the formula of Margulis–Russo directly using the increasing coupling between $\mathbb{P}_p$ and $\mathbb{P}_{p+\varepsilon}$, as $\varepsilon \to 0$.

**Definition.** Define $\theta_n(p)$ as the probability to connect to distance $n$:

$$\theta_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n),$$

where $\Lambda_n = [-n,n]^d$. For any $S \subset \mathbb{Z}^d$ subset of vertices define its edge boundary by

$$\Delta S := \{xy \in E(\mathbb{Z}^d) \, : \, x \in S, y \notin S\}.$$

**Exercise 16.** a) Let $S \subset \mathbb{Z}^d$ be a finite a subset of vertices containing 0 and let $z \in \mathbb{Z}^d \setminus S$. Show that the following estimate holds:[11]

$$\mathbb{P}_p(0 \leftrightarrow z) \le \sum_{xy \in \Delta S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x) \cdot \mathbb{P}_p(xy \text{ is open}) \cdot \mathbb{P}_p(y \leftrightarrow z).$$

b) Let $L > 1$ be integer and $S \subset \Lambda_{L-1}$ be a subset of vertices containing 0. Deduce from the above that, for every $N > L$,[12]

$$\theta_N(p) \le \theta_{N-L}(p) \cdot p \cdot \sum_{xy \in \Delta S} \mathbb{P}_p(0 \overset{S}{\leftrightarrow} x).$$

**Exercise 17.** Introduce a random variable $\mathscr{S}_n := \{z \in \Lambda_n : z \not\leftrightarrow \partial\Lambda_n\}$. Show that [13]

$$\theta_n'(p) = \tfrac{1}{(1-p)} \cdot \mathbb{E}_p\Bigg[\sum_{xy \in \Delta \mathscr{S}_n} \mathbb{P}_p(0 \overset{\mathscr{S}_n}{\longleftrightarrow} x)\Bigg].$$

**Exercise 18.** Number all edges in $E(\mathbb{Z}^d)$ as $\{e_i\}_{i \ge 1}$. For $n \in \mathbb{N}$, define a $\sigma$-algebra $\mathscr{F}_n := \sigma(e_1, \dots, e_n)$. Let $f$ be a bounded function on $\{0,1\}^{E(\mathbb{Z}^d)}$ measurable with respect to $\mathbb{P}_p$. Show that

$$\mathbb{E}_p(f \,|\, \mathscr{F}_n) \xrightarrow{n \to \infty} f \text{ a.s.}$$

**Exercise 19.** a) Call an event decreasing if its complement is increasing. Show that two decreasing events are positively correlated. What happens if one event is increasing and the other decreasing?

b) **Square-root trick:** for any increasing events $A, B$, show that

$$\max\{\mathbb{P}_p(A), \mathbb{P}_p(B)\} \ge 1 - \sqrt{1 - \mathbb{P}_p(A \cup B)}.$$

Extend the statement to $n \ge 2$ increasing events $A_1, \dots, A_n$:

$$\max_{1 \le 1 \le n}\{\mathbb{P}_p(A_i)\} \ge 1 - (1 - \mathbb{P}_p(\cup_{i=1}^n A_i))^{1/n}.$$

Consider a domain $Q_n := [-n, n] \times [-n, n-1]$. Define $H_n$ to be the event that there exists a horizontal crossing in $Q_n$:

$$H_n := \{\omega \in \{0,1\}^{E(\mathbb{Z}^2)} : \{-n\} \times [-n, n-1] \overset{Q_n}{\longleftrightarrow} \{n+1\} \times [-n, n-1].$$

The crucial property of $H_n$ is that we can compute its probability exactly when $p = \frac{1}{2}$ (see Exercise 21).

**Exercise 20.** Let $0 < p < 1$ be such that $\theta(p) > 0$.

a) Fix any $\varepsilon > 0$. Show that for $k$ large enough,

$$\mathbb{P}_p(\Lambda_k \not\leftrightarrow \infty) \le \varepsilon.$$

b) Take any $k$ as above. Show that, for any $n > k$,[14]

$$\mathbb{P}_p(\{\Lambda_k \overset{\Lambda_n}{\longleftrightarrow} \{-n\} \times [-n, n]\}) \ge 1 - \varepsilon^{\frac{1}{4}}.$$

c) Take any $k$ as above. Show that, for any $n > k$,

$$\mathbb{P}_p(\{(-1,0)\Lambda_k \overset{Q_n}{\longleftrightarrow} \{-n\} \times [-n, n]\} \cap \{(1,0) + \Lambda_k \overset{Q_n}{\longleftrightarrow} \{n\} \times [-n, n]\}) \ge 1 - 2\varepsilon^{\frac{1}{4}}.$$

d) Deduce that [15]

$$\mathbb{P}_p(H_n) \xrightarrow[n \to \infty]{} 1.$$

**Exercise 21.** a) Let $d = 2$ and $p = \frac{1}{2}$. Show that[16]

$$\mathbb{P}(H_n) = \frac{1}{2}.$$

b) Deduce from this and Exercise 20 that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.
c) Use sharpness and item a) to prove that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.
In conclusion, $p_c(\mathbb{Z}^2) = \frac{1}{2}$. This is a famous theorem of Kesten from 1980.

**Exercise 22. (Zhang's argument)** This is a seminal argument that gives alternative derivation of the fact that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$ by establishing directly non-coexistence of primal and dual infinite clusters.

a) Bound from the below the probability that the top of $\Lambda_n$ is connected to infinity:

$$\mathbb{P}_{\frac{1}{2}}([-n,n] \times \{n\} \xleftrightarrow{\mathbb{Z}^2 \setminus \Lambda_n} \infty) \geq 1 - \mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\leftrightarrow \infty)^{\frac{1}{4}}$$

b) Consider the event $A_n$ that both top and bottom sides of $\Lambda_n$ are connected to infinity by primal (open) edges in $\mathbb{Z}^2 \setminus \Lambda_n$, and that both left and right sides of $\Lambda_n$ are connected to infinity by dual edges in $\mathbb{Z}^2 \setminus \Lambda_n$. Show that

$$\mathbb{P}_{\frac{1}{2}}(A_n) \geq 1 - 4\mathbb{P}_{\frac{1}{2}}(\Lambda_n \not\leftrightarrow \infty)^{\frac{1}{4}}$$

c) Show that $\mathbb{P}_{\frac{1}{2}}(A_n) = 0$.
d) Deduce that $\theta(\frac{1}{2}) = 0$, whence $p_c \geq \frac{1}{2}$.

We say that a graph $G$ has exponential growth if there exists $c_{vg} > 0$ such that

$$|\Lambda_n| \geq e^{c_{vg} n}, \text{ for any } n.$$

**Exercise 23.** * The goal of this exercise is to show that $\theta(p_c) = 0$ for amenable transitive graphs of exponential growth. An example of such a graph is the Cayley of the Lamplighter group.

a) Assume that $\theta(p_c) > 0$. Show that, for some $c > 0$,

$$\forall x, y \in G \quad \mathbb{P}_{p_c}(x \leftrightarrow y) \geq c.$$

b) Consider $u_n(p) := \min_{x \in \partial\Lambda_n}\{\mathbb{P}_p(0 \leftrightarrow x)\}$. Show that[17]

$$u_{m+n}(p) \geq u_m(p) \cdot u_n(p).$$

c) Let $p < p_c$. Show that $\sum_{x \in G} \mathbb{P}_p(0 \leftrightarrow x) < \infty$.[18]
d) Use Items b) and c) to show that, for all $p < p_c$,

$$u_n(p) \leq e^{-c_{vg} n}.$$

e) Deduce that $\theta(p_c) = 0$.

**Exercise 24.** Consider a planar lattice $G$ embedded in a such a way that all its vertices have integer coordinates. Assume that $\mathbb{Z}^2$ acts transitively on $G$, that Bernoulli percolations on $G$ and on $G^*$ exhibit a sharp phase transition at some $p_c(G), p_c(G^*) \in (0,1)$, and that the infinite clusters are unique. Consider rectangles $R_{n,k} := [0,n] \times [0,k]$ and denote there sides by $\mathsf{Bottom}_{n,k}$, $\mathsf{Left}_{n,k}$, $\mathsf{Top}_{n,k}$, and $\mathsf{Right}_{n,k}$ (we will often omit the indices). Let $\mathscr{H}_{n,k}$ (resp. $\mathscr{V}n,k$) be an event of existence of a horizontal (resp. vertical) open crossing in $R_{n,k}$.

a) Show that $p_c(G) + p_c(G^*) \leq 1$.[19]

Below we assume that $p > p_c(G)$ and $p^* = 1 - p > p_c(G^*)$. We will obtain a contradiction, thus implying $p_c(G) + p_c(G^*) \geq 1$. Together with item a), this would prove that in fact $p_c(G) + p_c(G^*) = 1$.

For $s \in \mathbb{N}$ and $x \in \mathbb{Z}^2$, consider $S_x(s) := x + [-s, s]^2$. We will often omit $s$ from the notation (view it as a large nubmer that will be chosen at the end).

b) Consider the right half-plane $\mathbb{H} := [0, \infty) \times \mathbb{R}$, the parts of its boundary $\ell_+ := \{0\} \times [1, \infty)$ and $\ell_- := \{0\} \times (-\infty, -1]$ and $\ell := \ell_+ \cup \ell_-$. Show that, for any $m \in \mathbb{N}$, there exists a vertex $x = x(m)$ of $G$ such that the first coordinate of $x$ equals $m$ and[20]

$$\mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell_-) \geq \mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell_+) \quad \text{and} \quad \mathbb{P}_p(S_{x+(0,1)} \overset{\mathbb{H}}{\leftrightarrow} \ell_-) \leq \mathbb{P}_p(S_{x+(0,1)} \overset{\mathbb{H}}{\leftrightarrow} \ell_+)$$

c) For this choice of $x$, show that[21]

$$\mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell_-) \geq 1 - \sqrt{\mathbb{P}_p(S_x \overset{\mathbb{H}}{\not\leftrightarrow} \ell)} \quad \text{and} \quad \mathbb{P}_p(S_{x+(0,1)} \overset{\mathbb{H}}{\leftrightarrow} \ell_+) \geq 1 - \sqrt{\mathbb{P}_p(S_{x+(0,1)} \overset{\mathbb{H}}{\not\leftrightarrow} \ell)}.$$

d)* Deduce that $\mathbb{P}_p(0 \overset{*,\mathbb{H}}{\longleftrightarrow} \infty) = 0$. Similarly, $\mathbb{P}_p(0 \overset{\mathbb{H}}{\leftrightarrow} \infty) = 0$. [22]

e) Let $R := R_{n,k}$. Show that there exists $x = x(n,k) \in \mathbb{Z}^2$ and $x', x''$ adjacent to $x$ (in $\mathbb{Z}^2$!), such that, up to reflections of the rectangle, the following holds [23]

$$\begin{aligned} \mathbb{P}_p(S_x \overset{R}{\leftrightarrow} \mathsf{Bottom}) \geq \mathbb{P}_p(S_x \overset{R}{\leftrightarrow} \mathsf{Top}) \quad &\text{and} \quad \mathbb{P}_p(S_{x'} \overset{R}{\leftrightarrow} \mathsf{Bottom}) \leq \mathbb{P}_p(S_{x'} \overset{R}{\leftrightarrow} \mathsf{Top}) \\ \mathbb{P}_p(S_x \overset{R}{\leftrightarrow} \mathsf{Left}) \geq \mathbb{P}_p(S_x \overset{R}{\leftrightarrow} \mathsf{Right}) \quad &\text{and} \quad \mathbb{P}_p(S_{x''} \overset{R}{\leftrightarrow} \mathsf{Left}) \leq \mathbb{P}_p(S_{x''} \overset{R}{\leftrightarrow} \mathsf{Right}) \end{aligned}$$

f) Show that the distance of $x(n,k)$ to the boundary of $R(n,k)$ tends to infinity, as $n$ and $k$ tend to infinity.[24]

g)* Show that[25]

$$\max\{\mathbb{P}_p(\mathscr{V}_{n,k}), \mathbb{P}_p(\mathscr{H}_{n,k})\} \xrightarrow[n,k\to\infty]{} 1$$
$$\min\{\mathbb{P}_p(\mathscr{V}_{n,k}), \mathbb{P}_p(\mathscr{H}_{n,k})\} \xrightarrow[n,k\to\infty]{} 0.$$

h) Reach a contradiction and derive that $p_c(G) + p_c(G^*) = 1$.[26]

**Exercise 25.** a) Consider the site percolation on the triangular lattice: each vertex is open with probability $p$ independently of the others. Show that the proof of Kesten's theorem applies to this setting and implies that the critical point equals $1/2$. What should change in definition of $\varphi_p(s)$? How should we define the 'almost-square' $Q_n$ so that $\mathbb{P}_p(Q_n \text{ is crossed horizontally}) = 1/2$?

b) How can we formulate this percolation in terms of colorings of faces on the hexagonal lattice? Check that there is a two-to-one correspondence between site percolation configurations on the triangular lattice and even subgraphs of the hexagonal lattice. Show that these subgraphs can be split into non-intersecting simple cycles (that we will call *loops*).

**Exercise 26.** Let $p$ be the unique positive root of the equation $t^3 + 1 = 3t$ and let $p_c$ be the critical value of the bond percolation on the triangular lattice.

a) (**Star-triangle transformation**) Let $u, v, w$ be three vertices of some graph $G$ that are all connected by edges. Define $G'$ as a graph obtained from $G$ by removing the edges $uv, vw, wu$, adding a vertex $x$ and edges $xu, xv, xw$. Let $\mathbb{P}_t$ be the Bernoulli percolation with parameter $p$ on the edges of $G$. Define $\mathbb{P}'_t$ as the percolation on of $G'$, such that the parameter equals $1-t$ on edges $xu, xv, xw$ and it equals $p$ on all other edges.

Show that $\mathbb{P}_t$ and $\mathbb{P}'_p$ can be coupled in such a way that connections between different vertices of $G$ are the same.

b) Relate the hexagonal and the triangular lattice via the star-triangle transformation. Show the Bernoulli percolation on the edges of the triangular lattice with parameter $t$ exhibits an infinite cluster with a positive probability if and only if the Bernoulli percolation on the edges of the hexagonal lattice with parameter $1-t$ exhibits an infinite cluster.

c) Deduce that $p_c = t$.[27]

d) Find a degree three polynomial for the critical parameter of the Bernoulli percolation on the edges of the hexagonal lattice.

**Exercise 27.** Consider a simply connected domain $\Omega$ with a smooth boundary. Fix for distinct points $a, b, c, d \in \partial\Omega$. For any $\delta > 0$, define $(\Omega_\delta, a_\delta, b_\delta, c_\delta, d_\delta)$ to be a finite graph with four marked points: $\Omega_\delta$ equals $\delta\mathbb{Z}^2 \cap \Omega$, and $a_\delta, b_\delta, c_\delta, d_\delta$ are four distinct points on $\partial\Omega_\delta$ that are closest to $a, b, c, d$ (we assume here that $\Omega_\delta$ is connected and $\partial\Omega_\delta$ is a simple path). Prove that there exists $c = c(\Omega, a, b, c, d)$ such that, for any $\delta > 0$,

$$\mathbb{P}_{1/2}\left((a_\delta b_\delta \overset{\Omega_\delta}{\longleftrightarrow} (c_\delta d_\delta)\right) \geq c,$$

where $(a_\delta b_\delta)$ and $(c_\delta d_\delta)$ are the portions of $\partial\Omega_\delta$ from $a_\delta$ to $\delta$, and from $c_\delta$ to $\delta$, going counter-clockwise.[28]

**Exercise 28.** Let $\Omega$ be a domain on $\mathbb{C}$. For $\delta > 0$, let $\Omega_\delta$ equal $\Omega \cap \delta\mathbb{H}$ and $F^\delta$ be the parafermionic observable for percolation (defined in Lecture 7). Show that there exist $K, \alpha > 0$ such that, for any $u, v \in \Omega$ and any $\delta > 0$,[29]

$$|F^\delta(u^\delta) - F(v^\delta)| \leq K|u^\delta - v^\delta|^\alpha.$$

Deduce that $F^\delta$ can be extended to $\Omega$ continuously, so that it is a Hölder map. Conclude that this family of Hölder maps has a convergent subsequence.[30]

**Exercise 29.** a) Show that there exists $c > 0$ such that, for any $n \in \mathbb{N}$,[31]

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n) \leq c \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_{2n}).$$

b) Show that there exist $c, C > 0$ such that, for any $n \in \mathbb{N}$ and any $x \in \partial\Lambda_n$,

$$c \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n)^2 \leq \mathbb{P}_{1/2}(0 \leftrightarrow x) \leq C \cdot \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n)^2.$$

c) (Quasi-multiplicativity) Show that, for any $\alpha \in (0,1)$, there exists $c = c(\alpha) > 0$ such that, for any $N \in \mathbb{N}$ and any $1 \leq n \leq \alpha N$,

$$\frac{\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_N)}{\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n)} \leq \mathbb{P}_{1/2}(\Lambda_n \leftrightarrow \partial\Lambda_N) \leq c \cdot \frac{\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_N)}{\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n)}.$$

**Exercise 30.** Let $G = (V, E)$ be a finite graph. Pick any edge $e \in E$ and any configuration $\tau \in \{0,1\}^{E\setminus\{e\}}$. Fix any $q > 0$ and $p \in [0,1]$ and consider $\varphi^0_{G,p,q}$ (the FK percolation measure on $G$ with free boundary conditions). Show that the probability that the edge $e$ is open conditioned on the state of the other edges can be computed in the following way:

$$\varphi^0_{G,p,q}(\omega_e = 1 \,|\, \omega_{|E\setminus\{e\}} = \tau) = \begin{cases} p, & \text{if } x \overset{\tau}{\leftrightarrow} y, \\ \frac{p}{p+(1-p)q}, & \text{if } x \overset{\tau}{\not\leftrightarrow} y. \end{cases}$$

What is the answer in case of general boundary conditions?

**Exercise 31.** Consider the product $\sigma$-algebra $\mathscr{F}$ on $\{0,1\}^{E(\mathbb{Z}^d)}$, a finite set $E \subset E(\mathbb{Z}^d)$ and some configuration $\tau \in \{0,1\}^E$ Define the event $A_\tau \in \mathscr{F}$ by

$$A_\tau = \{\omega \in \{0,1\}^{E(\mathbb{Z}^d)} : \omega = \tau \text{ on } E\}.$$

Show that there exists two increasnig events $B_\tau, C_\tau$ such that $B_\tau \supset C_\tau$ and $A_\tau = B_\tau \setminus C_\tau$. Deduce that any event depending on finitely many edges can be written as a disjoint union of $B_\tau \setminus C_\tau$, for some $\tau$.

**Exercise 32. (Monotonicity properties of the FK percolation.)** Let $q \geq 1$ and $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^d$. Take any increasing event $A$ in $\{0,1\}^E$.

a) Let $p \leq p'$ and $q \geq q' \geq 1$. Show that[32]

$$\varphi^\xi_{G,p,q}(A) \leq \varphi^\xi_{G,p',q'}(A).$$

b) Let $\xi, \xi'$ be two partitions of $\partial G$ such that $\xi \leq \xi'$. By the latter we mean that $\xi'$ is a coarser partition than $\xi$: each element of the partition $\xi$ is entirely contained in some element of the partition $\xi'$. Show that

$$\varphi^\xi_{G,p,q}(A) \leq \varphi^{\xi'}_{G,p,q}(A).$$

c) Deduce that, for any boundary conditions $\xi$,

$$\varphi^0_{G,p,q}(A) \leq \varphi^\xi_{G,p,q}(A) \leq \varphi^1_{G,p,q}(A).$$

d) Consider any subgraph $H \subset G$. Show that

$$\varphi^0_{H,p,q}(A) \leq \varphi^0_{G,p,q}(A) \quad \text{and} \quad \varphi^1_{H,p,q}(A) \geq \varphi^1_{G,p,q}(A).$$

e) Let $G_n$ be a sequence of finite subgraphs increasing to $\mathbb{Z}^d$. Deduce from the previous points that the following weak limits exist: [33]

$$\varphi^0_{p,q} := \lim_{n\to\infty} \varphi^0_{G_n,p,q} \quad \text{and} \quad \varphi^1_{p,q} := \lim_{n\to\infty} \varphi^1_{G_n,p,q}.$$

Moreover, they are independent of the sequence $G_n$ and invariant to translations.

**Exercise 33.** Let $q \geq 1$ and $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^d$. Take any subgraph $H \subset G$ with the edge-set $F$. Consider any increasing events $A, B$ such that $A$ depends only on edges in $F$ and $B$ depends only on edges in $E \setminus F$.

a) Show that, for any boundary conditions $\xi$,

$$\varphi^0_{H,p,q}(A) \leq \varphi^\xi_{G,p,q}(A \,|\, B) \leq \varphi^1_{H,p,q}(A).$$

b) Consider dimension two. Assume that $B$ is the event that there exists a circuit of open edges that separates $H$ from $\partial G$. Show that

$$\varphi^1_{G,p,q}(A) \leq \varphi^\xi_{G,p,q}(A \,|\, B).$$

**Exercise 34.** a) **(Holley criterion.)** Show that, for strictly positive measures $\mu, \nu$, the condition we had in the domination Lemma is equivalent to that the following is satisfied for any two configurations $\omega, \omega'$:

$$\mu(\omega \wedge \omega')\nu(\omega \vee \omega') \geq \mu(\omega)\nu(\omega').$$

By $\omega \wedge \omega'$ we denote the configuration in which an edge is open if and only if it is open in both $\omega$ and $\omega'$; by $\omega \vee \omega'$ we denote the configuration in which an edge is open if and only if it is open in at least one of $\omega$ and $\omega'$.

b) **(FKG lattice condition.)** Let $\mu$ be a strictly positive probability measure on $\{0,1\}^E$. For distinct edges $e, f \in E$ and any configuration $\omega \in \{0,1\}^E$, denote by $\omega^{ef}$, $\omega_{ef}$, $\omega^e_f$, $\omega^f_e$ the four configurations that agree with $\omega$ on $E \setminus \{e,f\}$ and such that: $e$ is open in $\omega^{ef}$ and $\omega^e_f$, while $f$ is open in $\omega^{ef}$ and $\omega^f_e$. Assume the following condition is satisfied for any $e, f \in E$ and $\omega \in \{0,1\}^E$:

$$\mu(\omega^{ef})\mu(\omega_{ef}) \geq \mu(\omega^f_e)\mu(\omega^e_f).$$

Show that then $\mu$ is positively associated (satisfies the FKG inequality)[34]. Use this to check that the FK percolation satisfies the FKG inequality when $q \geq 1$.

**Exercise 35.** a) Show the claim from the proof of erogdicity of $\varphi^1_{p,q}$. More precisely, let $q \geq 1$ and assume that for any $n \in \mathbb{N}$ and any increasing events $A, B \in \{0.1\}^{\Lambda_n}$ we have mixing:

$$\lim_{\|x\| \to \infty} \varphi^1_{p,q}(A \cap \tau_x B) = \varphi^1_{p,q}(A)\varphi^1_{p,q}(B).$$

Derive that the same property holds for any events that depend on finitely many edges.

b) What should be changed in the proof to show mixing for $\varphi^0_{p,q}$?

**Exercise 36.** Derive ergodicity from mixing for $\varphi^1_{p,q}$.

**Exercise 37.** This exercise gives a different proof to the statement that the set of $p$ on which $\varphi^1_{p,q} \neq \varphi^1_{p,q}$ is at most countable number of $p$.

a) Show that $Z^1_{\Lambda_{2n},p,q} \geq \left(Z^1_{\Lambda_{2n},p,q}\right)^{2^d}$.[35]

b) For every $n$, define the function

$$f^1_n(p,q) := \frac{1}{|E(\Lambda_n)|} \log\left(Z^1_{\Lambda_n,p,q}\right).$$

Deduce from a) that, for every $q > 0, p \in [0,1]$, the sequence $f^1_{2^k}(p,q)$ has a limit. Denote the limiting function by $f(p,q)$. It is called the **free energy** of the model.

c) Define the function

$$f^0_n(p,q) := \frac{1}{|E(\Lambda_n)|} \log\left(Z^0_{\Lambda_n,p,q}\right).$$

Show that

$$\frac{f^1_n(p,q)}{f^0_n(p,q)} \xrightarrow[n\to\infty]{} 1.$$

Deduce that $f^0_{2^k}(p,q) \xrightarrow[k\to\infty]{} f(p,q)$.

d) Show that the left and right derivatives of

$$t \mapsto f\left(\frac{e^t}{1+e^t}, q\right) + \log(1+e^t)$$

are respectively $\varphi^1_{p,q}(\omega_e)$ and $\varphi^0_{p,q}(\omega_e)$, where $p = \frac{e^t}{1+e^t}$.[36]

e) Show that $p \mapsto f(p,q)$ is convex and therefore is not differentiable in at most countably many points. Conclude.[37]

Let $q \geq 1$. As in percolation, define the critical edge density in the FK percolation by

$$p_c(q) := \inf\{p \in [0,1] : \varphi^1_{p,q}(0 \leftrightarrow \infty) > 0\}.$$

**Exercise 38.** a) Show that the value of $p_c(q)$ remains the same if it is defined via free boundary conditions.

b) Show that in dimension one, $p_c(q) = 1$.

In the points below, we fix dimension $d \geq 2$ and $q \geq 1$. The aim is to show that in this case the phase transition is non-trivial.

c) Let $q \geq 1$ and $d \geq 2$. Show that $p_c(q) > 0$.[38]

d) Let $q \geq 1$ and $d \geq 2$. Show that $p_c(q) < 1$.[39]

**Exercise 39.** Fix $q \geq 1, p > p_c(q)$. Verify that the argument of Burton and Keane applies to show that $\varphi^1_{p,q}$-a.s. and $\varphi^0_{p,q}$-a.s., there exists a unique infinite cluster.

**Exercise 40.** a) Describe the Edwards–Sokal coupling in the reverse direction: given a spin configuration $\sigma \sim \mu^{\mathrm{f}}_{G,T,q}$ (Potts model), how to sample an edge configuration $\omega \sim \varphi^0_{G,p,q}$ (FK percolation model)? Express $T$ as a function of $p$.

b) Describe the measure on spin configurations obtained from an edge configuration $\omega \sim \varphi^1_{G,p,q}$ (FK-percolation model with wired boundary conditions) via the Edwards–Sokal coupling.

The measure defined in this exercise is called the Potts model is called the Potts model with *monochromatic boundary conditions*. The measure with color $i$ on the boundary is denoted by $\mu^i_{G,T,q}$.

**Exercise 41.** a) Let $\Omega_k$ be an increasing sequence of domains on $\mathbb{Z}^d$ that eventually exhausts the whole space. Fix $q \geq 1, T > 0$. Show that $\mu^f_{\Omega_k,T,q}$ has a weak limit that does not depend on the sequence of domains. We denote it by $\mu^{\mathrm{f}}_{T,q}$.

b) Show the same for the Potts model with monochromatic boundary conditions. The limit of $\mu^i_{\Omega_k,T,q}$ is denoted by $\mu^i_{T,q}$.

c) Show that measures $\mu^{\mathrm{f}}_{T,q}$, $\mu^1_{T,q}$, $\dots$, $\mu^q_{T,q}$ are translation invariant.

d) Show that the monochromatic measures $\mu^1_{T,q}$, $\dots$, $\mu^q_{T,q}$ are ergodic for any $T > 0$, and the free measure $\mu^{\mathrm{f}}_{T,q}$ is ergodic when $T > T_c$ and non-ergodic when $T < T_c$.

e) Describe the phase transition that occurs in the Potts model at $T = T(p_c)$.

**Exercise 42.** The goal of this exercise is to show the OSSS inequality for Bernoulli percolation:

$$\mathrm{Var}_p(f) \leq 2 \sum_{e \in E} \delta_e(f,T) \mathrm{Cov}_p(f, \omega_e),$$

where the variance and covariance are taken with respect to the Bernoulli percolation measure $\mathbb{P}_p$, $f : \{0,1\}^E \to [0,1]$ is an increasing function, $T = (e_1; \{\psi_t\})$ is any decision tree for $f$, $\delta_e(f,T) := \mathbb{P}_p(\exists t \leq \tau \ e = e_t)$ is the revealment probability.

The proof is similar to Lindeberg's proof of the Central Limit Theorem and is based on replacing bits in the percolation configuration by fresh Bernoulli random variables one by one. Let $\omega$ be distributed as $\mathbb{P}_p$. Applying to $\omega$ the decision tree $T$, we get ordering on the edges:

- $e_1$ is the starting edge;
- $e_{t+1} = \psi_t(e_1, \ldots, e_t; \omega_{e_1}, \ldots, \omega_{e_t}$.

Consider a percolation configuration $\eta$ distributed as $\mathbb{P}_p$ and independent of $\omega$. Given $t \in [0,n]$, define $\omega^t$ as follows: take $\omega$ and its values by $\eta$ at the first $t$ edges and after $\tau$. Formally:

$$\omega^t := (\eta_{e_1}, \ldots, \eta_{e_t}, \omega_{e_{t+1}}, \ldots, \omega_{e_\tau}, \eta_{e_{\tau+1}}, \ldots, \eta_{e_n}).$$

Below we denote by $\mathbb{P}$ the joint distribution of $\omega$ and $\eta$. By $\mathbb{E}$ we denote the expectation with respect to $\mathbb{P}$.

a) Show that $f(\omega^0) = f(\omega)$ and $f(\omega^n) = f(\eta)$.

b) By conditioning on $\omega$, deduce that

$$\mathrm{Var}_p(f) \leq \mathbb{E}\left[|f(\omega^0) - f(\omega^n)|\right].$$

c) Show that

$$\mathrm{Var}_p(f) \leq \sum_{t=1}^{n} \mathbb{E}\left[|f(\omega^t) - f(\omega^{t-1})| \cdot \mathbb{1}_{t \leq \tau}\right].$$

d) Fix $t$ and consider possible values of $e_t$. Using that $e_t$ and $\{t \geq \tau\}$ are measurable with respect to $\omega_{e_{[t-1]}} := (\omega_{e_1}, \ldots, \omega_{e_{t-1}})$, show that

$$\mathbb{E}\left[|f(\omega^t) - f(\omega^{t-1})|\mathbb{1}_{t\leq\tau}\right] = \sum_{e \in E} \mathbb{E}\left(\mathbb{E}\left(|f(\omega^t) - f(\omega^{t-1})| \,\middle|\, \omega_{e_{[t-1]}}\right) \cdot \mathbb{1}_{e_t = e, t \leq \tau}\right),$$

where the second expectation is taken with respect to the edges not in $e_{[t-1]}$.

e) Note that $\omega^t$ and $\omega^{t-1}$ are independent of $\omega_{e_{[t-1]}}$ and deduce, for every $e \in E$, the term on the right-hand side equals to

$$2p(1-p) \cdot \mathbb{E}(f(\omega[e,1]) - f(\omega[e,0])) \cdot \mathbb{P}(e_t = e, t \leq \tau),$$

where by $\omega[e,1]$ and $\omega[e,0]$ we denote the configurations that agrees with $\omega$ on $E \setminus \{e\}$ and such that the edge is open in $\omega[e,1]$ and closed in $\omega[e,0]$.

f) Show that

$$p(1-p)\mathbb{E}(f(\omega[e,1]) - f(\omega[e,0])) = \mathrm{Cov}_p(f, \omega_e).$$

Conclude.

**Exercise 43.** Fix dimension $d = 2$ and let $q \geq 1$. The goal of this exercise is to describe the dual FK percolation measure.

Let $G = (V,E) = [-n,n]^2$ (subgraph of $\mathbb{Z}^2$). Define $G^* = (V^*, E^*)$ (subgraph of the dual lattice $(\mathbb{Z}^2)^*$) as follows

- $V^*$ consists of all faces of $\mathbb{Z}^2$ that neighbor at least one edge of $G$;
- $E^*$ consists of all pairs of faces of $\mathbb{Z}^2$ that share an edge in $E$.

In particular, dual edges are in bijection with the primal edges: if $uv$ is a primal edge that separates faces $u^*$ and $v^*$, then we draw an edge $u^*v^*$.

Given a percolation configuration $\omega \in \{0,1\}^E$, we define the dual configuration $\omega^*$ in the usual way:

$$\omega^*_{e^*} = 1 - \omega_e.$$

Consider the FK percolation measure with free boundary conditions:

$$\varphi^0_{G,p,q}(\omega) = \frac{1}{Z^0_{G,p,q}} \cdot p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)}$$

a) Show that $o(\omega) = c(\omega^*)$ and $c(\omega) = o(\omega^*)$.

b) Show that

$$k((\omega^*)^1) - k(\omega) = o(\omega) + 1 - |V|,$$

where $(\omega^*)^1$ is obtained from $\omega^*$ by identifying (=merging) all boundary vertices.[40]

c) Show that, if $\omega$ has the law $\varphi^0_{G,p,q}$, then $\omega^*$ has the law $\varphi^1_{G^*,p^*,q}$, where $p^*$ satisfies

$$\frac{p}{(1-p)} \cdot \frac{p^*}{(1-p^*)} = q.$$

e) Deduce that the same holds also in the infinite-volume limit:

$$\text{if } \omega \sim \varphi^0_{p,q}, \text{ then } \omega^* \sim \varphi^1_{p^*,q},$$

where the second measure is taken on the dual lattice.

e) Show that on has $\varphi^0_{G,p_{\mathrm{sd}},q}(\omega) = \varphi^1_{G^*,p_{\mathrm{sd}},q}(\omega^*)$, where

$$p_{\mathrm{sd}} = p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{\sqrt{q}+1}.$$

f)* Describe the correspondence for general boundary conditions.

**Exercise 44.** The goal of this exercise is to show that on $\mathbb{Z}^2$, when $q \geq 1$, one has

$$p_c = p_{\mathrm{sd}}.$$

a) Assume that $p_c > p_{\mathrm{sd}}$. Apply sharpness of the phase transition at $p = p_{\mathrm{sd}}$ in the primal and in the dual model to show that sizes of clusters in $\omega$ (primal) and $\omega^*$ (dual) have exponential tails. Arrive at a contradiction.

b) Assume that $p_c < p_{\mathrm{sd}}$. Show that then[41]

$$\varphi^0_{p_{\mathrm{sd}},q}(\exists \text{ infinite cluster}) = \varphi^1_{p_{\mathrm{sd}},q}(\exists \text{ infinite cluster}) = 1.$$

Use the same for the dual measure and apply the Burton–Keane theorem to show

$$\varphi^0_{p_{\mathrm{sd}},q}(\exists \text{ unique infinite primal and dual clusters}) = 1.$$

Arrive at a contradiction via Zhang's argument (exercise 22).

c) Note that the above proves also that

$$\varphi^0_{p_c,q}(\exists \text{ infinite cluster}) = 0.$$

**Exercise 45.** Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^d$. The Potts model at $q = 2$ is called the Ising model and its configurations are viewed as possible ways to assign 1 or $-1$ to the vertices. Denote by $\mu^+_{G,T}$ the Ising measure with plus boundary conditions: all spins in $\partial G$ are fixed to by 1.

a) Show that

$$\mu^+_{G,T}(\sigma) \propto \exp\left[\frac{1}{T} \cdot \sum_{u\sim v} \sigma_u \sigma_v\right].$$

b) Show that, for any $x \in V$,

$$\mu^+_{G,T}(\sigma_x) = \varphi^1_{G,p,2}(x \leftrightarrow \partial G),$$

where $p$ as in the Edwards–Sokal coupling. Deduce that $\mu^+_{G,T}(\sigma_x) \geq 0$.

c) Use the Edwards–Sokal coupling to show that, for any $x, y, z, w \in V$,

$$\mu^+_{G,T}(\sigma_x\sigma_y\sigma_z\sigma_w) \geq \mu^+_{G,T}(\sigma_x\sigma_y)\mu^+_{G,T}(\sigma_z\sigma_w).$$

d)* **(First Griffiths' inequality.)** For $A \subset V$, use the notation $\sigma_A = \prod_{x\in A} \sigma_x$. Using the same ideas as above, show the first

$$\mu^+_{G,T}(\sigma_A) \geq 0.$$

e)* **(Second Griffiths' inequality.)** For any $A, B \subset V$, show that

$$\mu^+_{G,T}(\sigma_A\sigma_B) \geq \mu^+_{G,T}(\sigma_A)\mu^+_{G,T}(\sigma_B).$$

Note that this holds also for free boundary conditions.

# Hints

[1]Use (without a proof) that the Galton–Watson tree survives in and only if the expected number of offsprings for any given individual is strictly greater than 1.

[2]For any $k, n \in \mathbb{N}$, prove that

$$\limsup \tfrac{a_n}{n} \le \tfrac{a_k}{k}.$$

For this, divide $n$ by $k$ with a remainder $r$ and use sub-additivity.

[3]Use Dynkin's lemma

[4]Use approximation by cylinder events. Shift by a long enough vector to get events of disjoint support.

[5]Use Kolmogorov's zero-one law.

[6]For the first inequality: sample a Bernoulli site percolation configuration with parameter $p$ and define for it (in some way) a bond configuration. In the latter, each edge is open with probability $p$, but edges are *not independent*! The trick is then to show that this percolation model is *dominated* by the Bernoulli bond percolation of parameter $p$. To couple configurations, use uniform random variables (like in the lectures) but do this by *iteratively exploring* the cluster of the origin.

The second inequality can be shown similarly but you need to go from bonds to sites.

[7]Use percolation. To get a better bound, explore the cluster of $x$ step by step.

[8]Introduce a random variable $\mathscr{C}$ for the set of vertices connected to 0 in $S$ and sum over all possible values of $\mathscr{C}$. Use that on any path connecting 0 and $x$ there exists a vertex $y \in \partial S$.

[9]Use the coupling of percolation measures via uniform random variables.

[10]It's enough to generate cylinders.

[11]Use the same exploration strategy as in Exercise 11.

[12]Use that the distance from any point $y \in \partial S$ to $\partial \Lambda_N$ is at most $N - L$.

[13]Use the formula of Margulis–Russo. Then the trick is to force that the edge $e = xy$ is closed (recall $\mathrm{Piv}_e(A)$ does not depend on $e$!). Afterwards, you just need to assume $\mathscr{S}_n = S$ and sum over all possible values of $S$. The pivotality of $xy$ then reduces to the fact that $0 \overset{S}{\leftrightarrow} x$ and $xy \in \Delta S$. The sum over these values can be written as an expectation.

[14]Use the square-root trick.

[15]Use uniqueness

[16]Use symmetry

[17]Use Harris' (FKG) inequality.

[18]Adapt the proof of sharpness.

[19]Use sharpness and Borel-Cantelli to prove that the number of open circuits surrounding a given vertex is finite.

[20]When $x$ is very high, $\mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell) - \mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell_+) < \varepsilon$ (consider the distribution on top-most points on $\{0\} \times \mathbb{R}$ connected to $S_x$). Choose $\varepsilon$, so that $\mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \ell_+$ but $S_x \overset{\mathbb{H}}{\not\leftrightarrow} \ell_-) > \varepsilon$.

[21]Square-root trick

[22]First prove that $\mathbb{P}_p(S_x \overset{\mathbb{H}}{\leftrightarrow} \{0\} \times \mathbb{R}) \ge 1 - \varepsilon$, when $s$ is large enough (use uniqueness for this). Show that in fact this connection cannot only go through 0 (again, use uniqueness). Use item c) to connect $S_x$ to both $\ell_+$ and $\ell_-$ and then... use uniqueness once again.

[23]Color all points of $\mathbb{Z}^2$ into BL, BR, TL, TR, according to the sides that this points prefers (B is for Bottom, etc). Consider a path $\gamma$ on $\mathbb{Z}^2$ from the bottom-left to the top-right corner and look at the first time you get a point of TR type. If the previous point is BL, we are done. Assume it is TL. Look at the TL cluster of these points. Note that its external boundary must be TR, otherwise we are done. Consider the last point on $\gamma$ before this cluster and get a contradiction.

[24]Use item d).

[25]By the square-root trick, $S_x$ is connected to one of the sides with probability $1 - \varepsilon$. Same for $S_{x'}$ and $S_{x''}$. Consider which side is the best for which box. By the uniqueness, show that either Left or Bottom is connected to each of the three boxes with probability $1 - \varepsilon$. Conclude.

[26]Consider the largest $k$ such that $\mathbb{P}_p(\mathscr{H}_{n,k}) \le \mathbb{P}_p(\mathscr{V}_{n,k})$.

[27]Use duality.

[28]Use the RSW estimates.

[29]Use RSW.

[30]Use Arzela–Ascoli.

[31]Use RSW

[32]Use the domination Lemma from the lecture.

[33]For increasing events that depend on finitely many edges use monotonicity from d). By inclusion-exclusion, define the measure on all events that depend on finitely many edges — and then extend to the $\sigma$-algebra by Caratheodory's theorem. Invariance to translations follows if you shift the domains $G_n$.

[34]Show that $\mu(\omega \wedge \omega')\mu(\omega \vee \omega') \geq \mu(\omega)\mu(\omega')$ by induction in the number of disagreements between $\omega$ and $\omega'$. Then use the domination Lemma.

[35]Divide $\Lambda_{2n}$ into $2^d$ boxes of size $n$.

[36]Write derivative of $f_n^1(p,q)$ as an expectation — and then as an average probability that a particular edge is open. When $n$ is large, the measure is almost translation-invariant far from the boundary, and the edges next to the boundary have a small contribution (use amenability of $\mathbb{Z}^d$ for this, i.e. $|\partial\Lambda_n|/|\Lambda_n| \to 0$).

[37]Express $Z^1_{\Lambda_n,\tilde{p},q}/Z^1_{\Lambda_n,p,q}$ as expectation of some random variable with respect to $\varphi^1_{\Lambda_n,p,q}$ (see lecture 9). Take derivative in $\tilde{p}$ and show that this random variable is convex.

[38]Use monotonicity in $q$ to compare the FK percolation to the standard Bernoulli percolation.

[39]As in percolation: reduce to $d = 2$ by monotonicity, then use duality and the counting argument. To solve the exercise, you do not need to describe the distribution of the dual configuration precisely — but you can already try to find out what it is (we will discuss this next week).

[40]Use induction in $o(\omega)$ or Euler's formula.

[41]Use that $\varphi^0_{p,q} \neq \varphi^1_{p,q}$ at at most countably many $p$.