Lemma 3.

Let $f_n : [0, x_0] \to [0, M]$ be increasing, differentiable and satisfy:

- $f_n \xrightarrow[n\to\infty]{} f$
- $f_n' \geq \frac{n}{S_n} f_n$, for all n,

where $S_n = \sum_{k=0}^{n-1} f_k$.

Then there exists $x_1 \in [0, x_0]$, s.t.

(i) for $x < x_1$, there exists $c_x > 0$, s.t. for any n large enough

$$f_n(x) \leq \exp(-c_x n)$$

(ii) for $x > x_1$,

$$f(x) \geq x - x_1.$$

Proof (Lemma 3):

Define

$$x_1 := \inf\left\{x : \limsup_{n\to\infty} \frac{\log S_n(x)}{\log n} \geq 1\right\}$$

1) $x < x_1$

Fix $\delta > 0$. Take

$$x' := x - \delta, \quad x'' = x - 2\delta.$$

↳ • Step 1: stretched exponential decay.

There exists $\alpha > 0$, N, s.t.

$$\Sigma_n(x) \le n^{1-\alpha}, \text{ for all } n > N.$$

Since $f_n \nearrow$, we get for all $y < x$:

$$f_n'(y) \ge f_n(y)\, n^\alpha.$$

Integrate between x' and x:

$$\underbrace{\log f_n(x)}_{\log \le M} - \log f_n(x') \ge \delta \cdot n^\alpha$$

$$f_n(x') \le M \cdot \exp(-\delta n^\alpha)$$

↳ • Step 2: exponential decay.

By Step 1,

$$\Sigma_n(x') \le \Sigma < \infty, \text{ for all } n.$$

Since $\Sigma_n \nearrow$, we get for all $y < x'$:

$$f_n' \ge \frac{n}{\Sigma} \cdot f_n.$$

Integrate between x'' and x':

$$\underbrace{\log f_n(x')}_{\log \le M} - \log f_n(x'') \ge \delta \cdot n / \Sigma.$$

$$f_n(x'') \le M \cdot e^{-\delta n / \Sigma}$$

2) $x > x_1$.

Define $T_n := \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i}{i}$.

Differentiate:

$$T_n' = \frac{1}{\log n} \sum_{i=1}^{n} \frac{f_i'}{i} \geq \frac{1}{\log n} \cdot \sum_{i=1}^{n} \frac{f_i'}{\Sigma_i}.$$

Note that, for all i:

$$\frac{f_i}{\Sigma_i} = \frac{\Sigma_{i+1} - \Sigma_i}{\Sigma_i} \geq \int_{\Sigma_i}^{\Sigma_{i+1}} \frac{dt}{t}$$

$$= \log \Sigma_{i+1} - \log \Sigma_i$$

Thus, we get:

$$T_n' \geq \frac{1}{\log n} \cdot \left[\log \Sigma_{n+1} - \log \Sigma_1\right].$$

Integrate between $x' \in (x_1, x)$ and x:

$$\underbrace{T_n(x)}_{\to f(x)} - \underbrace{T_n(x')}_{\to f(x')} \geq (x - x') \cdot \frac{1}{\log n} \cdot \left[\log \Sigma_n(x') - \log M\right]$$

Thus,

$$f(x) - f(x') \geq (x - x') \cdot \overbrace{\limsup_{n\to\infty} \frac{\log \Sigma_n(x')}{\log n}}^{\geq 1}$$

Let $x' \to x_1^+$ and get

$$f(x) - f(x_1) \geq x - x_1$$

$\square$