Galton-Watson tree
(for a Branching process)

Offspring distribution: probability measure μ supported on $\mathbb{N} \cup \{0\}$

Define $p_k := \mu(\{k\})$.

Assume that $p_1 < 1$.

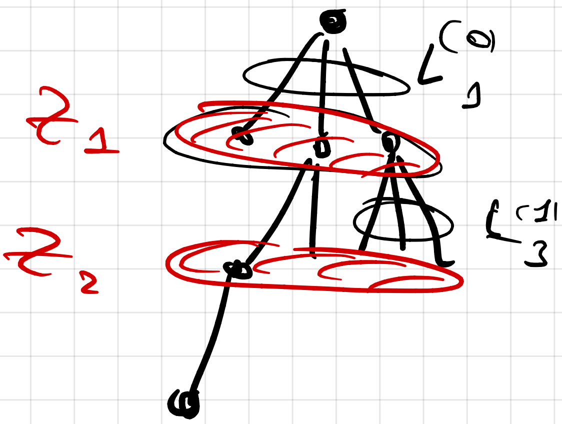
Consider $\{L_i^{(n)}\}_{n,i\in\mathbb{N}}$ - a family of iid rand var. having distribution μ.

Define $Z_0 := 1$ and:

$$Z_{n+1} = \begin{cases} \sum_{i=1}^{Z_n} L_i^{(n+1)} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases}$$

Z_n is: population size of the n-th generation in the Galton-Watson branching process.

One can draw a tree:



Generating functions:

for $s \in [0,1]$, $f_n(s) := \mathbb{E}(s^{Z_n})$.

then $f_0(s) = s$,

$f_1(s) = \mathbb{E}(s^L)$, where $L \sim \mu$.

$= \sum p_k s^k$.

and $f_{n+1}(s) = \mathbb{E}(s^{Z_{n+1}})$

$$= \mathbb{E}(\underbrace{\mathbb{E}(s^{Z_{n+1}} \mid Z_n)}_{\substack{\shortparallel \\ \mathbb{E}(s^{L_1^{(n+1)}} \cdots s^{L_{Z_n}^{(n+1)}} \mid Z_n) \\ \shortparallel \\ \prod_{i=1}^{Z_n} \mathbb{E}(s^{L_i^{(n+1)}}) \\ \shortparallel \\ (\mathbb{E}(s^L))^{Z_n}}})$$

$= f_n(f_1(s))$

②

Note that:

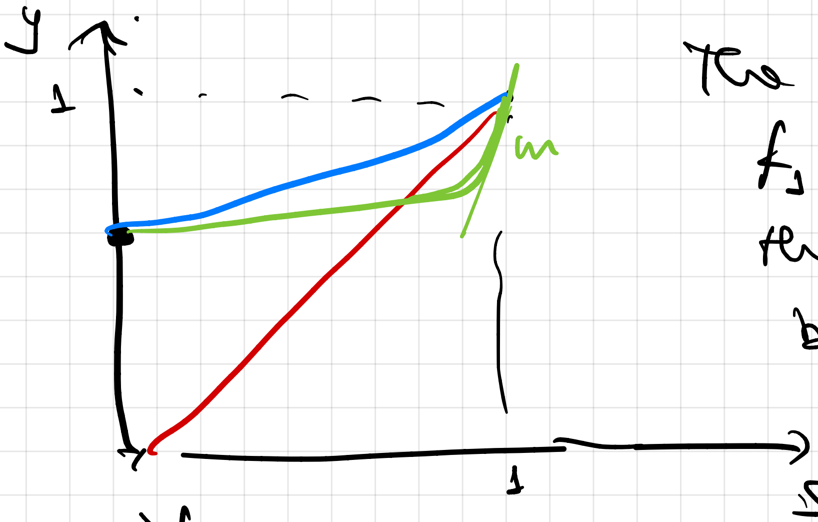
- f_1 is convex on $\mathbb{R}^+$: $f''(s) \geq 0$ for $s \geq 0$.
- $f_1'(1) = \mathbb{E}(L) =: m$

Let q be the smallest fixed point of f_1 in $[0,1]$.

Claim: $\underbrace{\lim_{n\to\infty} \mathbb{P}(Z_n = 0)}_{\text{probability of extinction}} = q$.

Proof: Indeed, $\mathbb{P}(Z_n = 0) = f_n(0)$.
$\leq f_n(q) = q$.

Also, $\lim_{n\to\infty} \mathbb{P}(Z_n = 0)$ is a fixed point of f_1. ☒



The question is whether $f_1(s)$ intersects the line $s = y$ before $s = 1$.

since $f_1 \nearrow$ and convex:

- if $m \leq 1$, then $q = 1$: extinction
- if $m > 1$, then $q < 1$: survival.