# Dynkin's lemma

Let $\mathcal{F}$ be a $\sigma$-algebra.

**Def**

- $\mathcal{P} \subset \mathcal{F}$ is a $\pi$-system if it is closed under $\cap$:
  $\forall A, B \in \mathcal{P} \quad A \cap B \in \mathcal{P}$.
- $\mathcal{D} \subset \mathcal{F}$ is a $d$-system if:
  (i) $\Omega \in \mathcal{D}$
  (ii) If $A \subseteq B$ and $A, B \in \mathcal{D}$, then $B \setminus A \in \mathcal{D}$
  (iii) If $A_n \subseteq A_{n+1}$ are in $\mathcal{D}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$

**Thm (Dynkin)**

Let $\mathcal{P}$ be a $\pi$-system.
Let $\mathcal{D}$ be a $d$-system.
Assume $\mathcal{P} \subseteq \mathcal{D}$.
Then $\sigma(\mathcal{P}) \subseteq \mathcal{D}$.

# Approximation by cylinder events.

We have indep. Bernoulli rand. var. $\omega_{ij} \sim Ber(p)$, for any $i \sim j$

Cylinder events:

$$\mathcal{C} := \bigcup_{n \geq 1} \sigma\left( (\omega_{ij})_{\substack{|i|,|j| \leq n \\ i \sim j}} \right)$$

Define our $\sigma$-algebra

$$\mathcal{F} := \sigma(\mathcal{C}).$$

Percolation measure:

$$\mathbb{P}_p := \prod_{\substack{i,j \in \mathbb{Z}^d \\ i \sim j}} Ber(p).$$

Formally:

there is a unique probab. meas. on $\mathcal{F}$, s.t. $S_0, S_1 \subset E^d$ - finite disjoint, we have:

$$\mathbb{P}_p(\omega|_{S_0} = 0, \omega|_{S_1} = 1) = p^{|S_1|} \cdot (1-p)^{|S_0|}$$

[Kolmogorov's extension theorem]

## Lemma

All events in $\mathcal{F}$ are approximable by cylinder events:

$\forall A \in \mathcal{F}\ \exists \{B_n\}$ - events in $\mathcal{C}$:

$$\mathbb{P}_p(A \triangle B_n) \longrightarrow 0$$

**Proof:**

We want to use Dynkin's lemma.

- $\pi$-system: $\mathcal{C}$.
- $d$-system: $\underbrace{\text{approximable events}}_{\mathcal{D}}$.

Clearly, $\mathcal{C} \subseteq \mathcal{D}$.

So by Dynkin's lemma:

$$\underbrace{\sigma(\mathcal{C})}_{\mathcal{F}} \subseteq \mathcal{D}$$

Remains to check that $\mathcal{D}$ is a $d$-system

(i) $\Omega \in \mathcal{D}$ – trivial

(ii) $A, B \in \mathcal{D}$ and $A \subseteq B$.
Then, $B \setminus A \in \mathcal{D}$.

Proof:

Consider $\varepsilon > 0$ and $A', B' \in \mathcal{C}$
s.t. $\mathbb{P}(A \triangle A') < \varepsilon$
$\mathbb{P}(B \triangle B') < \varepsilon$.

Then,

$$\mathbb{P}_\rho((B \setminus A) \triangle (B' \setminus A'))$$

$$= \mathbb{P}(\underbrace{(B \setminus A) \setminus (B' \setminus A')}_{\subseteq (B \setminus B') \cup (A' \setminus A)}) + \mathbb{P}(\underbrace{(B' \setminus A') \setminus (B \setminus A)}_{\subseteq (B' \setminus B) \cup (A \setminus A')})$$

$$\leq \mathbb{P}(B \setminus B') + \mathbb{P}(A' \setminus A) + \mathbb{P}(B' \setminus B) + \mathbb{P}(A \setminus A')$$

$$< 2\varepsilon. \quad \blacksquare$$

(iii) $A_n \in \mathcal{D}$, $A_n \subseteq A_{n+1}$.
Then $\bigcup_n A_n \in \mathcal{D}$.

Proof: Fix $\varepsilon > 0$. $A = \bigcup A_n$

Take $n$ s.t. $\mathbb{P}(A \setminus A_n) < \varepsilon$

Take $B \in \mathcal{C}$: $\mathbb{P}(A_n \triangle B) < \varepsilon$.

Then $\mathbb{P}(A \triangle B) \leq \varepsilon + \mathbb{P}(A_n \triangle B)$
$\leq 2\varepsilon. \quad \blacksquare$