Step 3: If p is the **continuity** point of $p \mapsto \varphi^1_{p,q}(\omega_e = 1)$, then

$$\varphi^0_{p,q}(\omega_e = 1) = \varphi^1_{p,q}(\omega_e = 1).$$

<u>Proof</u>:

Fix p and $\tilde{p} < p$. Take

$$a := \varphi^0_{p,q}(\omega_e = 1)$$

$$b := \varphi^1_{\tilde{p},q}(\omega_e = 1) \xrightarrow[\tilde{p} \to p]{} \varphi^1_{p,q}(\omega_e = 1)$$

So it's enough to show: $b \leq a$.

For $n \geq 1$, we have

$$\varphi^0_{\Lambda_n,p,q}(\underbrace{\#\text{open edges in } \omega}_{O(\omega)})$$

$$= \sum_{e \in E(\Lambda_n)} \underbrace{\varphi^0_{\Lambda_n,p,q}(\omega_e = 1)}_{\leq \varphi^0_{p,q}(\omega_e = 1) = a} \leq a \cdot |E_n|$$

Hence:

$$\varphi^0_{\Lambda_n,p,q}(O(\omega) > |E|) < a$$

Moreover, for any $\varepsilon \leq 1 - a$,

$$\varphi^0_{\Lambda_n,p,q}(O(\omega) > (a+\varepsilon)|E|) < 1 - \varepsilon,$$

since $(a+\varepsilon)(1-\varepsilon) \geq a$.

Then,

$$(\star) \quad \varphi^{0}_{\Lambda_n,p,q}(O(\omega) \leq (a+\varepsilon)|E_n|) \geq \varepsilon$$

Similarly,

$$\varphi^1_{\Lambda_n,\tilde p,q}(C(\omega)) \le (1-b)|E_n|$$

As above $(1-b \mapsto a)$, we get

$$\varphi^1_{\Lambda_n,\tilde p,q}(C(\omega) \le (1-b+\varepsilon)|E_n|) \ge \varepsilon,$$

for any $\varepsilon \le b$.

Subtract both sides from $|E_n|$:

$(**)$ $$\varepsilon \le \varphi^1_{\Lambda_n,\tilde p,q}(O(\omega) \ge (b-\varepsilon)|E_n|)$$

$k(\omega) \le k(\omega') + |\partial\Lambda_n|$

$\Downarrow$

$Z^0 \le Z^1 \cdot q^{|\partial\Lambda_n|}$

$$\le q^{|\partial\Lambda_n|}\varphi^0_{\Lambda_n,\tilde p,q}(O(\omega) \ge (b-\varepsilon)|E_n|)$$

Note that, for a rand. var. X:

$$\sum_{\omega\in\{0,1\}^E} X(\omega)\,\tilde p^{O(\omega)}(1-\tilde p)^{C(\omega)}q^{k(\omega)}$$

$$=(1-\tilde p)^{|E|}\sum_\omega X(\omega)\left(\frac{\tilde p}{1-\tilde p}\right)^{O(\omega)}q^{k(\omega)}$$

$$=(1-\tilde p)^{|E|}\sum_\omega X(\omega)\Bigg(\underbrace{\frac{\tilde p}{1-\tilde p}\cdot\frac{1-p}{p}}_{\lambda<1}\Bigg)^{O(\omega)}\left(\frac{p}{1-p}\right)^{O(\omega)}q^{k(\omega)}$$

$$=\left(\frac{1-\tilde p}{1-p}\right)^{|E|}\sum_\omega X(\omega)\,\lambda^{O(\omega)}p^{O(\omega)}(1-p)^{C(\omega)}q^{k(\omega)}$$

Compute this for
- $X = \mathbb{1}_{O(\omega)\ge(b-\varepsilon)|E_n|}$
- $X = 1$

and ~~take~~ the ratio.

We get:

$$\varphi^{0}_{\Lambda_n,\tilde{p},q}\left(O(\omega)\geq(b-\varepsilon)|E_n|\right)$$

$$= \frac{\varphi^{0}_{\Lambda_n,p,q}\left(\lambda^{O(\omega)}\cdot \mathbb{1}_{O(\omega)\geq(b-\varepsilon)|E_n|}\right)}{\varphi^{0}_{\Lambda_n,p,q}\left(\lambda^{O(\omega)}\right)}$$

$$\leq \frac{\varphi^{0}_{\Lambda_n,p,q}\left(\lambda^{O(\omega)}\cdot \mathbb{1}_{O(\omega)\geq(b-\varepsilon)|E_n|}\right)}{\varphi^{0}_{\Lambda_n,p,q}\left(\lambda^{O(\omega)}\cdot \mathbb{1}_{O(\omega)\leq(a+\varepsilon)|E_n|}\right)}$$

$$\overset{(*)}{\leq} \frac{1}{\varepsilon}\cdot\frac{\lambda^{(b-\varepsilon)|E_n|}}{\lambda^{(a+\varepsilon)|E_n|}} = \lambda^{(b-a-2\varepsilon)|E_n|}\,\frac{1}{\varepsilon}$$

Substitute into $(**)$:

$$\varepsilon^2 \leq q^{|\partial\Lambda_n|}\cdot\lambda^{(b-a-2\varepsilon)|E_n|}$$

Recall that $\lambda<1$ and $\frac{|\partial\Lambda_n|}{|\Lambda_n|}\xrightarrow[n\to\infty]{}0$.

Taking large n, we get

$$b\leq a+2\varepsilon.$$

Since ε was arbitrary:

$$b\leq a$$

$\blacksquare$