Matrix Models, 2D Gravity & non-critical Strings (summary notes)

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1 Ribbon Graphs and Topological expansion

1.1 Warmup. Scalar case

Gaussin (probability) measure on $\mathbb{R}$:

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{1}$$

Normalization :

$$\int_{-\infty}^{\infty} d\mu(x) = 1 \tag{2}$$

Definition : For any (integrable with meausre $d\mu$) function $f : \mathbb{R} \to \mathbb{C}$ we define its expectaion value $\langle f \rangle$ as :

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x) d\mu(x) \tag{3}$$

For integer n :

$$\langle x^n \rangle = \begin{cases} 0 & n = 2k+1 \\ (n-1)!! & n = 2k \end{cases} \tag{4}$$

Exercise Check it using integration by parts.

Thats of course a very simple integral, however, let's calculate it in a lengthy but instructive way, introducing auxiliary "current" j :

$$\langle x^{2k} \rangle = \int_{-\infty}^{\infty} dx x^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{d^{2k}}{dj^{2k}} \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+jx} \Big|_{j=0} \tag{5}$$

$$= \frac{d^{2k}}{dj^{2k}} e^{\frac{j^2}{2}} \Big|_{j=0} = \frac{1}{2^k k!} \frac{d^{2k}}{dj^{2k}} j^{2k} \tag{6}$$

One of course can immediately take derivatives and get $(2k)!/2^k k! = (2k-1)!!$, however, we will go further and split $2k$ derivatives into k pairs and distribute them over $2k$ j's. Now let's notice that the derivatives in any pair are interchangeable as well as the k pairs themselves, what gives the factor $2^k k!$ precisely cancelling the denominator and finally we get :

$$\frac{1}{2^k k!} \frac{d^{2k}}{dj^{2k}} j^{2k} = \sum_{\text{all pairings}} \prod_{\text{pairs}} < x^2 >= (2k-1)!! \tag{7}$$

This result can be interpreted as a simple combinatorial counting : the first pair can be chosen in $2k-1$ ways, second - $2k-3$ etc. This can be represented graphically as a pairings of half-edges attached to $2k$ ordered dots (see.Fig.1.a) or, alternatively, to one vertex of valency $2k$ (Fig.1.b.).

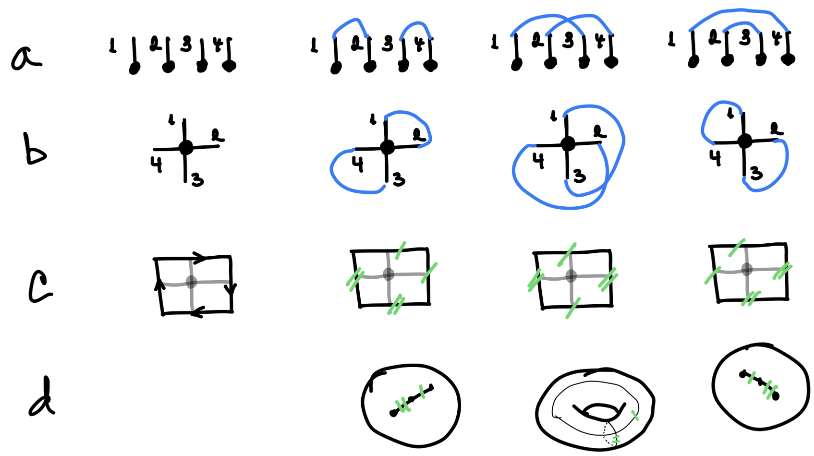

Figure 1: Graphical illustration for $\langle x^{2k} \rangle$ with $k=2$. a) pairings of $2k$ ordered dots b) pairing of the ordered half-edges of the $2k$-vertex c) oriented $2k$-gon dual to $2k$-vertex. d) different ways to glue oriented $2k$-gon. In the particular case of square we get two spheres and one torus.

Another combinatorial/graphical interpretation comes from the dual picture, namely let's go from the vertex of valency $2k$ to the dual graph - oriented $2k$-gon (Fig.1.c.). An pairing in this case corresponds to the gluing (wrt the orientation) of $2k$ edges and each particular pairing results into the oriented Riemann surface of a certain topology. In particular case of $k=2$ we get a square and three different gluings give two spheres and one torus (Fig.1.d.).

Exercise Find the number Cat_k of ways to glue the sphere from a polygon with $2k$ sides.

Hint : All spherical gluings come from the pairings without selfintersections, like this:

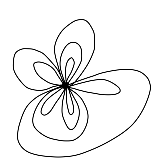

Figure 2: Typical polygon's gluing giving spherical topology

Show that $\text{Cat}_{k+1} = \sum_{i=0}^{k} \text{Cat}_i \text{Cat}_{k-i}$. Introduce generation function $f(x) = \sum_{k=0}^{\infty} \text{Cat}_k x^k$ and using recursion relation write an equation for $f(x)$. Solve it. Expand around $x = 0$ and get closed form for Cat_k. What is the radius of convergence? Check that Cat_k coincide with Catalan numbers.

Now let's consider the following exponential generation function for $\langle x^{4k} \rangle$

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2} - gx^4} \underset{\Re(g)>0}{=} \frac{e^{\frac{1}{32g}}}{4\sqrt{\pi g}} K_{\frac{1}{4}}(\frac{1}{32g}) \tag{8}$$

Formal expansion gives :

$$I(g) \to I_{series}(g) = \sum_{k=0}^{\infty} (-g)^k \frac{\langle x^{4k} \rangle}{k!} = \sum_{k=0}^{\infty} (-g)^k \frac{(4k-1)!!}{k!} \tag{9}$$

asymptotic series with zero radius of convergence. That's a typical situation, very often generation functions for combinatorial objects have just asymptotic expansion. The same phenomena ubiquitously appear in theoretical physics : typical perturbation series in Quantum Mechanics/QFT/Strings are asymptotic. Particularly $I(g)$ can be interpreted as partition function of quartic anharmonic oscillator with coupling constant g. We will discuss the relation between the resummation of asymptotic series and convergent integrals later, now let's just do the following numerical exercise :

Exercise *(for Mathematica)*

The best approximation one can get from the partial resummation $S_m(g) = \sum_{k=0}^{m} a_k g^k$ of asymptotic series is up to the term k^* with the minimal absolute value $|a_{k^*}|$. The last can be found using Stirling's formula :

$$\log |a_k| \sim k(\log 16n - 1 - \log \frac{1}{g}),$$
$$\frac{d}{dk} \log |a_k| = 0|_{k=k^*}, \quad \to \quad k^* = \frac{1}{16g}$$

so the optimal truncation is $k^* = \frac{1}{16g}$ and the error can be estimated as the absolute value of the minimal term : $Error(g) = I(g) - S_{k^*}(g) \sim e^{-k^*} = e^{-\frac{1}{16g}}$

- Reproduce pictures from Fig.3.

- Does $e^{-\frac{1}{16g}}$ give a correct scale for $Error(g)$? Is it getting better if one keeps subleading terms in Stirling's formula?

- Make (analytically/using Mathematica) Borel summation of $I_{series}(g) = \sum a_k g^k$: $I_{\mathcal{B}}(t) = \mathcal{B}[I_{series}(g)](t) = \sum b_k t^k$, where $b_k = a_k/k!$. What is the pole & cut structure of $I_{\mathcal{B}}(t)$ as a function of complex variable t? Does it have any singularity on the real positive line? Make (numerically) inverse Borel transform $f(g) = \int_0^{\infty} dt I_{\mathcal{B}}(tg) e^{-t}$, compare $f(g)$ with original convergent integral $I(g)$.

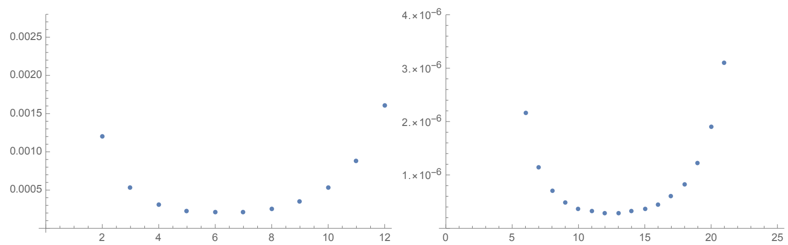

Figure 3: Partial sums $I(g) - S_k(g)$. Left: $g = 0.01$, $k^* = 6.25$. Right : $g = 0.005$, $k^* = 12.5$

1.2 Multidimensional case. Wick theorem

Let's consider the following gaussian measure:

$$d\mu(\mathbf{x}) = \frac{1}{\mathcal{Z}_0} e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}} d^N\mathbf{x} \tag{10}$$

where $\mathbf{x} = \{x_i\} \in \mathbb{R}^N$ is N-dimensional vector, A is real symmetric invertable matrix and $\mathcal{Z}_0 = \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}} d^N\mathbf{x}$.

Normalization:

$$\int\limits_{\mathbb{R}^N} d\mu(\mathbf{x}) = 1 \tag{11}$$

Let's calculate constant $\mathcal{Z}_0$. First, we introduce new variables $\mathbf{y} = O^{-1}\mathbf{x}$ where $O \in O(N)$ is orthogonal matrix diagonalising A : $A = O\Lambda O^{-1}$, $\Lambda = \text{diag}\{\lambda_i\}$. Then using $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T\Lambda\mathbf{y}$ and $d^N\mathbf{x} = d^N\mathbf{y}$ we get :

$$\begin{aligned}\mathcal{Z}_0 &= \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}\mathbf{x}^T A\mathbf{x}} d^N\mathbf{x} = \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}\sum\limits_{i=1}^{N}\lambda_i y_i^2} d^N\mathbf{y} \\ &= \prod_{i=1}^{N} \int\limits_{\mathbb{R}} e^{-\frac{1}{2}\lambda_i y_i^2} dy_i = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}}\end{aligned} \tag{12}$$

Exercise For those who familiar with grassmann variables . It is instructive to compare bosonic gaussian integral with its grassmann analogue. Check :

1) $\int e^{-\theta^T A\eta} d^n\theta d^n\eta = \det A$ for a complex $n \times n$ matrix A and grassmann vector variables $\theta \& \eta$

2) $\int e^{-\frac{1}{2}\theta^T A\theta} d^{2k}\theta = \text{Pf}A$ for a complex skew-symmetric $2k \times 2k$ matrix A, and PfA is the Pfaffian of A : $(\text{Pf}A)^2 = \det A$.

For the odd size $(2k+1) \times (2k+1)$: $\int e^{-\frac{1}{2}\theta^T A\theta} d^{2k+1}\theta = 0$

Now let's calculate two-point function:

$$\langle x_l x_m \rangle = \int\limits_{\mathbb{R}^N} x_l x_m d\mu(\mathbf{x}) = \frac{1}{\mathcal{Z}_0} \frac{\partial^2}{\partial j_l \partial j_m} \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x} + \mathbf{j}^T \mathbf{x}} \bigg|_{\mathbf{j}=0} \tag{13}$$

where similarly to one-dimensional case we introduced auxiliary "current" $\mathbf{j} \in \mathbb{R}^N$. Completing the square in the power of exponent we get:

$$\frac{1}{\mathcal{Z}_0} \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x} + \mathbf{j}^T \mathbf{x}} = \frac{1}{\mathcal{Z}_0} \int\limits_{\mathbb{R}^N} e^{-\frac{1}{2}(\mathbf{x} - A^{-1}\mathbf{j})^T A (\mathbf{x} - A^{-1}\mathbf{j}) + \mathbf{j}^T \mathbf{x} + \frac{1}{2}\mathbf{j}^T A^{-1} \mathbf{j}} = e^{\frac{1}{2}\mathbf{j}^T A^{-1} \mathbf{j}} \tag{14}$$

So we can write:

$$\langle x_l x_m \rangle = \frac{\partial^2}{\partial j_l \partial j_m} e^{\frac{1}{2}\mathbf{j}^T A^{-1} \mathbf{j}} \bigg|_{\mathbf{j}=0} =$$
$$\frac{\partial^2}{\partial j_l \partial j_m} \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\mathbf{j}^T A^{-1} \mathbf{j})^k \bigg|_{\mathbf{j}=0} = \frac{\partial^2}{\partial j_l \partial j_m} \frac{1}{2} \mathbf{j}^T A^{-1} \mathbf{j} = A^{-1}_{lm} = B_{lm} \tag{15}$$

or introducing the inverse matrix $B = A^{-1}$:

$$\langle x_l x_m \rangle = B_{lm} \tag{16}$$

Similarly for $\langle x_{i_1} ... x_{i_4} \rangle$ we get :

$$\begin{aligned} \langle x_{i_1} ... x_{i_4} \rangle &= \frac{\partial^4}{\partial j_{i_1} ... \partial j_{i_4}} e^{\frac{1}{2}\mathbf{j}^T A^{-1} \mathbf{j}} \bigg|_{\mathbf{j}=0} = \frac{\partial^4}{\partial j_{i_1} ... \partial j_{i_4}} \frac{1}{2^2 2!} (\mathbf{j}^T A^{-1} \mathbf{j})^2 \\ &= B_{i_1 i_2} B_{i_3 i_4} + B_{i_1 i_3} B_{i_2 i_4} + B_{i_1 i_4} B_{i_2 i_3} \end{aligned} \tag{17}$$

Now let's turn to the general case $\langle x_{i_1} ... x_{i_n} \rangle$. Obviously it is nonzero only when n is even.

Exercise Why?

In the general case $n = 2k$ we get :

$$\begin{aligned} \langle x_{i_1} ... x_{i_n} \rangle &= \frac{\partial^l}{\partial j_{i_1} ... \partial j_{i_n}} e^{\frac{1}{2}\mathbf{j}^T A^{-1} \mathbf{j}} \bigg|_{\mathbf{j}=0} \\ &= \frac{\partial^l}{\partial j_{i_1} ... \partial j_{i_n}} \frac{1}{2^k k!} (\mathbf{j}^T A^{-1} \mathbf{j})^k \end{aligned} \tag{18}$$

Again, just acting by all derivatives and cancelling $\frac{1}{2^k k!}$ factor (2^k reflects the symmetry of B_{ij} while $k!$ - permutations of k factors B_{ij}) we arrive to the Wick's theorem:

Wick's theorem:

$$\langle x_{i_1}...x_{i_n}\rangle = \sum_{\text{all pairings of}\{i_1,...,i_n\}} \prod_{\text{pairs}(i_\alpha,i_\beta)} B_{i_\alpha,i_\beta} \tag{19}$$

Comment: This statement can be generalised as :

$$\langle f_{i_1}...f_{i_n}\rangle = \sum_{\text{all pairings of}\{i_1,...,i_n\}} \prod_{\text{pairs}(i_\alpha,i_\beta)} < f_{i_\alpha} f_{i_\beta} > \tag{20}$$

where $f_i(\mathbf{x})$ are any linear functions of $\mathbf{x}$. Indeed, let's notice that the left and right sides are linear wrt f_i so it should be proved just for monomials what is exactly the statement of Wick's theorem (we remind that $< x_i x_j >= B_{ij}$)

Wick's theorem is still valid in the case when some of indexes coincide (compare with one-variable case from the previous section!), so we can reformulate it in the following way :

$$\langle x_{i_1}^{p_1}...x_{i_n}^{p_n}\rangle = \sum_{\text{all graphs } G \text{ with } n \text{ vertices of valencies } \{p_i\}} \prod_{\text{pairs}(i_\alpha,i_\beta)} B_{i_\alpha,i_\beta}, \tag{21}$$

where $\sum p_i$ is even, otherwise the expectation value is zero.

Example: $\langle x_1^3 x_2^5\rangle$

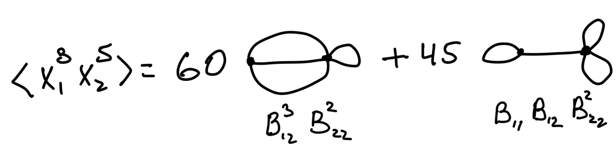

Figure 4: Two different graphs contributing to $\langle x_1^3 x_2^5\rangle$

As one can see from the above example, 105 terms actually groups into the two groups of pairings related to only two graphs. The corresponding prefactors can be obtained with help of *orbit-stabilizer theorem*

Orbit-Stabilizer Theorem

Let G be a finite group that acts on a finite set X, let O_x be the orbit of an element $x \in X$ and S_x - its stabilizer. Then $|O_x||S_x| = |G|$.

Now let's define the graph's automorphism :

Defenintion : Graph automorphism (symmetry) An automorphism(symmetry) of a graph $G = (V, E)$ is a permutation σ of the vertex set V, such that the pair of vertices (u, v) forms an edge if and only if the pair $(\sigma(u), \sigma(v))$ also forms an edge. The set of all automorphisms of the given graph form the group $\mathrm{Aut}(G)$.

Comment : In the case of colored graphs the vertices of the given color should be mapped to the vertices of the same color.

Now let's back to the correlator $\langle x_{i_1}^{p_1} ... x_{i_n}^{p_n} \rangle$ (without loss of generality we can assume that all indexes $\{i_k\}$ are different) and let's introduce a group $\mathrm{Aut} = \mathrm{Aut}(\{p_i\})$ of all authomorphisms of the vertices $\{p_i\}$, it consists only of the product of permutations of the half-edges inside of the vertices, so its rank is $|\mathrm{Aut}| = \prod p_i!$. This group naturally acts on the set of all labelled graphs appearing in (21) . For a given labelled graph G its stabilizer is exactly $Aut(G)$ while the number of times it appears in the sum is the length of this orbit $|O_G|$ under the action of Aut, so using orbit-stabilizer theorem we get:

$$\frac{1}{|\mathrm{Aut}|} < x_{i_1}^{p_1} ... x_{i_n}^{p_n} >= \sum_{\text{topologically different graphs } G} \frac{1}{|\mathrm{Aut(G)}|} \prod_{(i_\alpha, i_\beta) \text{ edges of G}} B_{i_\alpha, i_\beta}, \tag{22}$$

Exercise : reproduce factors from the previous example $\langle x_1^3 x_2^5 \rangle$

1.3 Matrix Integrals

1.3.1 The case of complex Hermitian matrices with Gaussin potential

Let's consider the following matrix integral:

$$\mathcal{Z}_0 = \int_{H_N} dM e^{-\frac{N}{2} \mathrm{tr} M^2} \tag{23}$$

where the integration goes over the set H_N of Hermitian matrices $N \times N$ and the Lebesgue measure is chosen as

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\Re M_{ij} d\Im M_{ij}, \tag{24}$$

where each of $N^2 = N + 2\frac{N(N-1)}{2}$ components runs along real line $(-\infty, \infty)$. Direct calculation gives:

$$\mathcal{Z}_0 = 2^{\frac{N}{2}} \left(\frac{\pi}{N}\right)^{\frac{N^2}{2}} \tag{25}$$

Exercise : Check!

The normalized measure is $d\mu(M) = \frac{1}{\mathcal{Z}_0} e^{-\frac{N}{2}\mathrm{tr}M^2} dM$. Expectation value :

$$\langle f(M)\rangle = \int\limits_{H_N} d\mu(M) f(M) \tag{26}$$

The power $\mathrm{tr}M^2$ is a real symmetric quadratic form wrt N^2 real variables $\{M_{ii}, \Re M_{ij}, \Im M_{ij}\}$ so one can use (16) to get propagator (2-point correlator) :

$$\langle M_{ij} M_{kl}\rangle = \frac{1}{N}\delta_{il}\delta_{jk} \tag{27}$$

Exersice Check it.

Exersice Get the same result using auxillary "matrix current" J_{ij} similarly to what we did in the scalar and vector case.

Matrix elements M_{ij} are linear combinations of real variables $\{M_{ii}, \Re M_{ij}, \Im M_{ij}\}$ so we can use Wick's theorem (20) to calculate the expectation value of the product of matrix elements as a sum over pairings. For example let's consider $\mathrm{tr}M^4$

$$\langle N\mathrm{tr}M^4\rangle = N\sum_{i,j,k,l}\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle =$$

$$N\sum(\langle M_{ij}M_{jk}\rangle\langle M_{kl}M_{li}\rangle + \langle M_{ij}M_{kl}\rangle\langle M_{jk}M_{li}\rangle + \langle M_{ij}M_{li}\rangle\langle M_{jk}M_{kl}\rangle)$$

$$= \frac{1}{N}\sum(\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}) = \frac{1}{N}(N^3 + N + N^3) = 2N^2 + 1$$

Let's represent this calculation graphically :

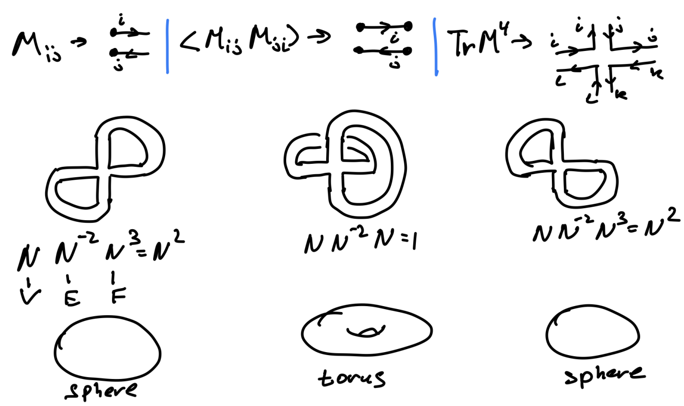

Figure 5: First line : graphical representation of the propagator and vertex. Second line : three possible contractions for $\mathrm{tr}M^4$ Third line : corresponding topologies.

Exercise Those who wants to get more experience with such calculations can perform the similar analysis (direct calculation + graphical representation) for $\langle (N\mathrm{tr}M^3)^2\rangle$.

Exercise *for Mathematica* Write Mathematica function(s) *WContr* which automatically performs Wick's contractions in $\langle \mathrm{tr}M^{p_1}...\mathrm{tr}M^{p_k}\rangle$. The input can be in the form of the product of several M_{ij}: WContr[M[i,j]....M[k,l]], e.g for $\langle \mathrm{tr}M^4\rangle$: WContr[M[i1,i2]M[i2,i3]M[i3,i4]M[i4,i1]] . Or even it can be the list of $\{p_k\}$. Using this function check yourself in the previous exercise.

In the general case of the correlator built of n traces $N\mathrm{tr}M^p$:

$$\langle (N\mathrm{tr}M^{p_1})(N\mathrm{tr}M^{p_2})....(N\mathrm{tr}M^{p_n})\rangle \tag{28}$$

($\{p_i\}$ are not necessary different) one get a sum over ribbon graphs built from the gluing of n ribbon vertices and the contribution from the given ribbon graph G is equal to $N^{V-E+F} = N^{\chi(G)}$, where $\chi(G)$ is Euler characteristic of the graph G. Indeed each vertex brings N, edge (propagator) - N^{-1} and face (closed lines in double-line notations, corresponding to the product of delta-Kronekers) - N. So we arrive to the Topological Expansion for correlators:

Theorem. Topological expansion for correlators

$$\frac{1}{\prod\limits_j n_j!}\langle \prod_{k=1}^{n} N\frac{\mathrm{tr}M^{p_k}}{p_k}\rangle = \sum_G \frac{N^{\chi(G)}}{|Aut(G)|} \tag{29}$$

where sum goes over graphs G with n vertices of valency $\{p_k\}$. The factor p_k in the denominator $\frac{\mathrm{tr}M^{p_k}}{p_k}$ is the rank of the symmetry group of the ribbon vertex of valency p_k because only cyclic permutations are allowed. The rank of the symmetry group of all vertices (vertex of valency p_k is taken n_k times) is $|Aut| = \prod\limits_j n_j! p_j^{n_j}$

Alternatively one can think in terms of the dual graphs (=glued surfaces) Σ.

$$\frac{1}{|Aut|}\langle \prod_{k=1}^{n} N\mathrm{tr}M^{p_k}\rangle = \sum_\Sigma \frac{N^{\chi(\Sigma)}}{|Aut(\Sigma)|} \tag{30}$$

where sum goes over surfaces Σ with n polygons of size $\{p_k\}$. Let's mention that the Euler characteristic of the graph and its dual coincide : $\chi(G) = \chi(\Sigma)$ because the numbers of vertices and faces interchage.

Literature comment Reader interested in the formal definition of ribbon graphs/maps, group of authomorphisms etc can take a look into [2, 3] for further details.

1.3.2 The case of complex Hermitian matrices with general potential

Let's consider the general potential $V(M) = \frac{1}{2}M^2 - \sum\limits_{k=3}^{\infty} \frac{t_k}{k} M^k$.

Theorem. Topological expansion for partition function

$$\frac{1}{\mathcal{Z}_0} \int\limits_{H_N} dM e^{-N\mathrm{tr}V(M)} = \sum_G \frac{N^{\chi(G)}}{|Aut(G)|} \prod_k t_k^{\#k-veritces} \tag{31}$$

where sum goes over all ribbon graphs G built of vertices with nonzero $\{t_k\}$.

Proof As usually we understand this matrix integral just as a formal expansion and sign "=" means equality between two formal series.

Using the following expansion :

$$e^{\sum_k a_k} = \sum_{\{n_k\}} \prod_k \frac{a_k^{n_k}}{n_k!} \tag{32}$$

we can write :

$$e^{\sum\limits_{k=3}^{\infty} \frac{t_k}{k} N\mathrm{tr}M^k} = \sum_{\{n_k\}} \prod_i \frac{1}{n_k!} \left(\frac{t_k}{k} N\mathrm{tr}M^k \right)^{n_k} \tag{33}$$

and

$$\frac{1}{\mathcal{Z}_0} \int\limits_{H_N} dM e^{-N\mathrm{tr}V(M)} = \sum_{\{n_k\}} \langle \prod_k \frac{1}{n_k!} \left(\frac{t_k}{k} N\mathrm{tr}M^k \right)^{n_k} \rangle_0 \tag{34}$$

where index 0 in the averaging $\langle ... \rangle_0$ reminds us that it is made wrt gaussian quadratic potential as in the previous subsection. Let's choose one particular term from the rhs numerated by the finite list $\{n_k\}$: $\langle \prod_k \frac{1}{n_k!} \left(\frac{t_k}{k} N\mathrm{tr}M^k \right)^{n_k} \rangle_0$. Assembling all numbers in one prefactor we get : $\frac{1}{\prod_k n_k! k^{n_k}}$ which is exactly $\frac{1}{|Aut|}$ where Aut is the automorphism's group of the collection $\{n_k\}$ of ribbon verteces (the vertex of valency k is taken n_k times). So we can write :

$$\langle \prod_k \frac{1}{n_k!} \left(\frac{t_k}{k} N\mathrm{tr}M^k \right)^{n_k} \rangle_0 = \frac{\prod_i t_i^{n_i}}{|Aut|} \langle \prod_k \left(N\mathrm{tr}M^k \right)^{n_k} \rangle_0 \tag{35}$$

Using (29) we can write :

$$\frac{\prod_i t_i^{n_i}}{|Aut|} \langle \prod_k \left(N\mathrm{tr}M^k \right)^{n_k} \rangle_0 = \prod_i t_i^{n_i} \sum_G \frac{N^{\chi(G)}}{|Aut(G)|} \tag{36}$$

Plugging it into the (34) we get (31) ■

Let's stress that the sum in (31) goes over all graphs , connected and not. The sum only over connected graphs gives "free energy" - logarithm of the partition function :

Theorem. Topological expansion for free energy

$$\log\left(\frac{1}{\mathcal{Z}_0}\int\limits_{H_N} dMe^{-N\mathrm{tr}V(M)}\right)=\sum_{G^c}\frac{N^{\chi(G^c)}}{|Aut(G^c)|}\prod_k t_k^{\#k-veritces} \tag{37}$$

where now the sum goes only over connected graphs G^c what we stressed by the extra label "c".

Proof . We will exponentiate the right hand side and show that it matches the rhs of (31). Namely we will establish one to one correspondence between the terms from two formal series and check that their contributions are equal. Let's exponentiate the rhs of (37) using (32):

$$\exp\left(\sum_{G^c}\frac{N^{\chi(G^c)}}{|Aut(G^c)|}\prod_k t_k^{\#k-veritces}\right)=\sum_{\{n_i\}}\prod_i\frac{1}{n_i!}\left(\frac{N^{\chi(G_i^c)}}{|Aut(G_i^c)|}\prod_k^{G_i^c} t_k^{\#k-veritces}\right)^{n_i} \tag{38}$$

where index i numerates[1] all connected finite graphs G_i^c and the product $\prod_k^{G_i^c} t_k^{\#k-veritces}$ goes over vertices of G_i^c what we stressed by the extra label "G_i^c". Let's take a certain term uniquely numerated by $\{n_i\}$, it is in one to one correspondence with a graph $G=\sqcup_i n_i G_i^c$ - disjoint union of n_i copies of G_i^c. Using additivity of Euler characteristic on disjoint graphs:

$$\chi(\sqcup_i n_i G_i^c)=\sum_i n_i\chi(G_i^c) \tag{39}$$

we can write:

$$\prod_i\left(N^{\chi(G_i^c)}\right)^{n_i}=N^{\chi(G)}. \tag{40}$$

Now let's notice that

$$|Aut(G)|=\prod_i n_i!|Aut(G_i^c)|^{n_i} \tag{41}$$

and

$$\prod_k^G t_k^{\#k-veritces}=\prod_i\left(\prod_k^{G_i^c} t_k^{\#k-veritces}\right)^{n_i} \tag{42}$$

Finally combining (40),(41),(42) we get:

$$\prod_i\frac{1}{n_i!}\left(\frac{N^{\chi(G_i^c)}}{|Aut(G_i^c)|}\prod_k^{G_i^c} t_k^{\#k-veritces}\right)^{n_i}=\frac{N^{\chi(G)}}{|Aut(G)|}\prod_k t_k^{\#k-veritces} \tag{43}$$

[1]E.g. one can think about i as a multi-index specifiying numbers of vertices of different valency and an extra index numerating the connected graphs built of them. Explicit choice of i doesn't matter for us.

what is exactly the term from the rhs of (31) corresponding to the graph G. ■

Now let's consider the following correlation function of n marked polygons[2] $\{p_i\}$ $\{p_i\}$ with marked edges :

$$\frac{1}{\mathcal{Z}_0}\int\limits_{H_N} dMN\mathrm{tr}M^{p_1}...N\mathrm{tr}M^{p_n}e^{-N\mathrm{tr}V(M)} = \tag{44}$$

$$\sum_{n_3,n_4,...}\langle N\mathrm{tr}M^{p_1}...N\mathrm{tr}M^{p_n}\left(\frac{Nt_3}{3}\right)^{n_3}\frac{(\mathrm{tr}M^3)^{n_3}}{n_3!}\left(\frac{Nt_4}{4}\right)^{n_4}\frac{(\mathrm{tr}M^4)^{n_4}}{n_4!}....\rangle_0 \tag{45}$$

Each term here is a correlation function built of n marked polygons with marked edges and any number of (unmarked) polygons coming from potential $V(M)$. Marked polygons with marked edges cann't be permuted or rotated so thier group of automorphisms is trivial (compare with prefactor $\frac{1}{\prod p_i^{n_i} n_i!)}$ in (30) where polygons are unmarked) , however, unmarked polygons from $V(M)$ come with a standard prefactor $\prod_j \frac{1}{j^{n_j} n_j!}$. So we get:

$$\frac{1}{\mathcal{Z}_0}\int\limits_{H_N} dMN\mathrm{tr}M^{p_1}...N\mathrm{tr}M^{p_n}e^{-N\mathrm{tr}V(M)} = \tag{46}$$

$$= \sum_{\Sigma\in\tilde{\Sigma}_{\mu_1,...,\mu_n}}\frac{N^{\chi(\Sigma)}}{|Aut(\Sigma)|}\prod_k t_k^{\#k-gons} \tag{47}$$

where the sum goes over all discrete surfaces, connected or not, with n marked faces (with marked edges) and any number of unmarked faces (see Fig.6).

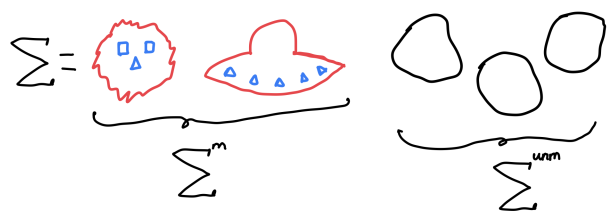

Figure 6: Graphical representation of an term from (46). Red connected components contain at least one of the marked faces colored blue. Black connected components are built entirely from the unmarked polygons coming from $V(M)$

Let's write an surface $\Sigma \in \tilde{\Sigma}_{\mu_1,...,\mu_n}$ as a disjoint union of all its marked Σ^m and all its unmarked Σ^{unm} components :

$$\Sigma = \Sigma^m \sqcup \Sigma^{unm} \tag{48}$$

[2]now we prefer to work with dual graphs

so we get:

$$\frac{N^{\chi(\Sigma)}}{|Aut(\Sigma)|}\prod_k t_k^{\#k-gons} = \left(\frac{N^{\chi(\Sigma^m)}}{|Aut(\Sigma^m)|}\prod_{\text{k-gons}\in\Sigma^m_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons}\right)\left(\frac{N^{\chi(\Sigma^{unm})}}{|Aut(\Sigma^{unm})|}\prod_{\text{k-gons}\in\Sigma^{unm}_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons}\right) \tag{49}$$

This factorization immediately leads to the factorization of the whole series:

$$\sum_{\Sigma\in\tilde{\Sigma}_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma)}}{|Aut(\Sigma)|}\prod_k t_k^{\#k-gons} =$$
$$\left(\sum_{\Sigma^m\in\Sigma^m_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma^m)}}{|Aut(\Sigma^m)|}\prod_{\text{k-gons}\in\Sigma^m_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons}\right)\left(\sum_{\Sigma^{unm}\in\Sigma^{unm}_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma^{unm})}}{|Aut(\Sigma^{unm})|}\prod_{\text{k-gons}\in\Sigma^{unm}_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons}\right) \tag{50}$$

The sum in the second brackets goes over all possible unmarked surfaces giving precisely the partition function (31):

$$\left(\sum_{\Sigma^{unm}\in\Sigma^{unm}_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma^{unm})}}{|Aut(\Sigma^{unm})|}\prod_{\text{k-gons}\in\Sigma^{unm}_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons}\right) = \frac{1}{\mathcal{Z}_0}\int_{H_N} dMe^{-N\mathrm{tr}V(M)} \tag{51}$$

So we can divide on it, redefine the definition of the measure and expectation value :

$$\langle f(M)\rangle = \frac{\int_{H_N} dMf(M)e^{-N\mathrm{tr}V(M)}}{\int_{H_N} dMe^{-N\mathrm{tr}V(M)}} \tag{52}$$

and write the following formula for the correlation function of n marked polygons :

$$\langle N\mathrm{tr}M^{p_1}...N\mathrm{tr}M^{p_n}\rangle = \sum_{\Sigma^m\in\Sigma^m_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma^m)}}{|Aut(\Sigma^m)|}\prod_{\text{k-gons}\in\Sigma^m_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons} \tag{53}$$

Finally let's devide both sides by N^n. On the right hand side it gives us $N^{\chi(\Sigma^m)-n}$ or writing the defenition of Euler characteristic explicitly : $\chi(\Sigma^m)-n = V-E+(F-n)$. We grouped F and n together and it can be interpreted as erasing of n faces or in other words one can think about marked faces as holes in the surface. Let's notice that the Euler characteristic of a surface Σ_n with n holes is exactly $\chi(\Sigma_n) = \chi(\Sigma) - n$. So we can present the final expression for the correlation function:

$$\langle \mathrm{tr}M^{p_1}...\mathrm{tr}M^{p_n}\rangle = \sum_{\Sigma\in\Sigma_{\mu_1,\dots,\mu_n}}\frac{N^{\chi(\Sigma)}}{|Aut(\Sigma)|}\prod_{\text{k-gons}\in\Sigma_{\mu_1,\dots,\mu_n}} t_k^{\#k-gons} \tag{54}$$

where sum goes over all discrete surfaces (not necessary connected) whose connected components contain at least one marked face with marked edges and where marked faces are counted as holes for computing the Euler characteristic.

Defenition. Cumulants

Let's introduce the set of trace products numerated by Young tableaux:

$$p_\mu(M) = \mathrm{tr}M^{\mu_1}\mathrm{tr}M^{\mu_2}...\mathrm{tr}M^{\mu_l}, \qquad \mu = (\mu_1, ..., \mu_l): \quad \mu_1 \geq \mu_2 \geq .... \geq \mu_l > 0 \tag{55}$$

Now we define the cumulant $\langle p_\mu(M)\rangle_c$ through the following relations:

$$\langle p_{\mu_1,...,\mu_l}\rangle = \sum_{k=1}^{l} \sum_{\sqcup I_j = \{\mu_1,...,\mu_l\}} \prod_{j=1}^{k} \langle p_{I_j}\rangle_c \tag{56}$$

First three lines written explicitly :

$$\langle p_{(\mu_1)}\rangle = \langle p_{(\mu_1)}\rangle_c, \tag{57}$$

$$\langle p_{(\mu_1,\mu_2)}\rangle = \langle p_{(\mu_1,\mu_2)}\rangle_c + \langle p_{(\mu_1)}\rangle_c\langle p_{(\mu_2)}\rangle_c, \tag{58}$$

$$\begin{aligned}\langle p_{(\mu_1,\mu_2,\mu_3)}\rangle &= \langle p_{(\mu_1,\mu_2,\mu_3)}\rangle_c + \langle p_{(\mu_1)}\rangle_c\langle p_{(\mu_2,\mu_3)}\rangle_c + \langle p_{(\mu_2)}\rangle_c\langle p_{(\mu_1,\mu_3)}\rangle_c \\ &+ \langle p_{(\mu_3)}\rangle_c\langle p_{(\mu_1,\mu_2)}\rangle_c + \langle p_{(\mu_1)}\rangle_c\langle p_{(\mu_2)}\rangle_c\langle p_{(\mu_3)}\rangle_c\end{aligned} \tag{59}$$

Equations (56) can be solved one by one giving explicit form for cumulants.

Statement

$$\langle \mathrm{tr}M^{\mu_1}...\mathrm{tr}M^{\mu_n}\rangle_c = \sum_{\Sigma^c \in \Sigma^c_{\mu_1,...,\mu_n}} \frac{N^{\chi(\Sigma^c)}}{|Aut(\Sigma^c)|} \prod_{\text{k-gons}\in\Sigma^c_{\mu_1,...,\mu_n}} t_k^{\#k-gons} \tag{60}$$

where sum goes only over $\Sigma^c_{\mu_1,...,\mu_n}$ - connected components of $\Sigma_{\mu_1,...,\mu_n}$.

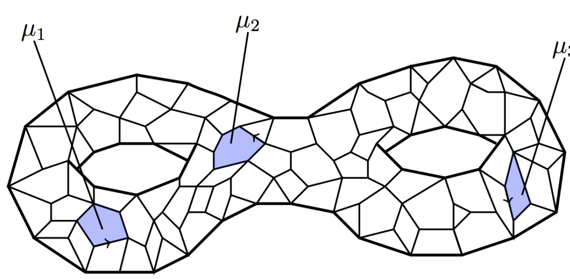

Figure 7: An discrete surface contributing to $\langle p_\mu\rangle_c$ with $\mu = \{\mu_1, \mu_2, \mu_3\} = \{5, 6, 4\}$. Picture is taken from [1]

It is useful to introduce the generation function for correlation functions $\langle p_\mu(M)\rangle$, $\langle p_\mu(M)\rangle_c$:

Defenition. Resolvent

$$W(x_1, ..., x_n) = \langle \mathrm{tr}\frac{1}{x_1 - M} ... \mathrm{tr}\frac{1}{x_n - M} \rangle \tag{61}$$

Correlation function $\langle \mathrm{tr}M^{\mu_1}\mathrm{tr}M^{\mu_2}...\mathrm{tr}M^{\mu_n}\rangle$ can be obtained from the formal $x_i \to \infty$ expansion of W, as a coefficient in front of $x_1^{-1-\mu_1}...x_n^{-1-\mu_n}$ or alternatively , using dif form[3]:

$$\langle p_\mu(M)\rangle = (-1)^n \operatorname*{Res}_{x_1\to\infty} ... \operatorname*{Res}_{x_n\to\infty} x_1^{\mu_1}...x_n^{\mu_n} W(x_1, ..., x_n)dx_1...dx_n \tag{62}$$

1.4 Other ensembles and Model building. Brief overview

Bicolored graphs and Ising model

Gas of colored loops

Six-vertex model on random graphs

2 Coulomb gas method

2.1 Reduction to eigenvalues

Let's consider the convergent matrix integral with polynomial potential of even degree

$$\frac{1}{\mathcal{Z}_0}\int_{H_N} dMe^{-N\mathrm{tr}V(M)}, \qquad V(M) = \frac{1}{2}M^2 - \sum_{k=3}^{d}\frac{t_k}{k}M^k, \qquad t_d < 0 \tag{63}$$

Spectral decomposition for Hermitian matrices :

$$M = U\Lambda U^{-1} \tag{64}$$

where $U \in U(N)$ and $\Lambda = \mathrm{diag}\{\lambda_1, ..., \lambda_N\} \in \mathbb{R}^N$. All traces $\mathrm{tr}M^k = \mathrm{tr}\Lambda^k = \sum_{i=1}^{N}\lambda_i^k$ reduce to eigenvalues and the potential $V(M) = \sum_{i=1}^{N}V(\lambda_i)$ is invariant under the conjugation with U. The measure dM is also invariant : $dUMU^{-1} = dM$.

Indeed ,one can think about the change of variables as the linear action of $U_\mathbb{R}\otimes U_\mathbb{R}^{-1}$: $M_\mathbb{R} \to U_\mathbb{R}M_\mathbb{R}U_\mathbb{R}^{-1}$ where subscript $\mathbb{R}$ indicates that we represent complex matrix $A_\mathbb{C}$ of size $n\times n$ as a real matrix[4] $A_\mathbb{R} = \begin{pmatrix}\mathrm{Re}\ A_\mathbb{C} & -\mathrm{Im}\ A_\mathbb{C}\\ \mathrm{Im}\ A_\mathbb{C} & \mathrm{Re}\ A_\mathbb{C}\end{pmatrix}$

[3]We remind that to calculate the residue of the differential form at infinity one should change the variable $x \to 1/z$, so $dx = -1/z^2dz$

[4]we need the real-matrix representation because one should think about the conjugation UMU^{-1} as a change of real coordinates.

Using that $\det A \otimes B = (\det A)^n (\det B)^n$ and $\det A_{\mathbb{R}} = |\det A_{\mathbb{C}}|^2$ we conculde that Jacobian $\det U_{\mathbb{R}} \otimes U_{\mathbb{R}}^{-1} = 1$ and $dUMU^{-1} = dM$. So it looks natural to change the coordinates from $\{M_{ii}, \Re M_{ij}, \Im M_{ij}\}$ to $\{\Lambda, U\}$.

One can get the volume form in the new coordinates rewriting the metric. First let's notice that our original volume form (24) can be read off from the following metric:

$$\mathrm{tr}(dM)^2 = \sum_{i=1}^{N} (dM_{ii})^2 + 2\sum_{i<j}^{N} \left((d\Re M_{ij})^2 + (d\Im M_{ij})^2 \right) \tag{65}$$

Indeed, the metric is diagonal and the corresponding volume form :

$$d\mathrm{vol}(M) = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} 2d\Re M_{ij} d\Im M_{ij} \tag{66}$$

coincides with (24) up to a factor $2^{\frac{N(N-1)}{2}}$. So we can just rewrite the metric in new coordinates and get the Jacobian. With this end in view let's calculate δM using spectral decomposition:

$$\begin{aligned} \delta M = d(U\Lambda U^{-1}) &= d(U)\Lambda U^{-1} + Ud(\Lambda)U^{-1} + U\Lambda d(U^{-1}) \\ &= d(U)\Lambda U^{-1} + Ud(\Lambda)U^{-1} - U\Lambda U^{-1} d(U) U^{-1} \\ &= U\left(d\Lambda + U^{-1}d(U)\Lambda - \Lambda U^{-1} d(U)\right) U^{-1} = ULU^{-1} \end{aligned} \tag{67}$$

where we introduced $L = d\Lambda + [\Omega, \Lambda]$, $\Omega = U^{-1}dU$ and used[5] $d(U^{-1}) = -U^{-1}d(U)U^{-1}$ in the second line. Let's mention that Ω is skew-hermitian matrix $\Omega_{ij} = -\bar{\Omega}_{ji}$

So we get:

$$\begin{aligned} \mathrm{tr}(dM)^2 = \mathrm{tr}L^2 &= (\lambda_i \delta_{ij} + (\lambda_j - \lambda_i)\Omega_{ij})(\lambda_j \delta_{ji} + (\lambda_i - \lambda_j)\Omega_{ji}) \\ &= \sum_{i=1}^{N} d^2\Lambda_{ii} + 2\sum_{i<j} (\lambda_i - \lambda_j)^2 |\Omega_{ij}|^2 \end{aligned} \tag{68}$$

Introducing $\Omega_{ij} = dx_{ij} + idy_{ij}$ we can write $|\Omega_{ij}|^2 = d^2x_{ij} + d^2y_{ij}$ and the corresponding volume form :

$$d\mathrm{vol}(M) = \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} d\lambda_i \underbrace{\prod_{i<j} 2dx_{ij}dy_{ij}}_{dU_{Haar}} \tag{69}$$

So we get for the partition function :

[5]that follows from $0 = d(Id) = dUU^{-1} = dUU^{-1} + Ud(U^{-1})$

$$\mathcal{Z} = \int d\Lambda dU_{Haar} \Delta(\Lambda)^2 e^{-N\mathrm{tr}V(\Lambda)}, \tag{70}$$

where we introduced notation $\Delta(\Lambda)$ for the Vandermonde determinant $\Delta(\Lambda) = \prod\limits_{i<j}(\lambda_i - \lambda_j)$. In this formula we didn't specify the integration domain, to do so let's revisit the spectral decomposition (64). Actually this decomposition is not unique, namely one can i) multiply matrix U by any diagonal matrix $U(1)^N$ from the right and ii) permute eigenvalues and the corresponding lines&rows in U so it gives us the following set of parameters :

$$H_N \cong \left(\mathbb{R}^N \times U(N)/U(1)^N\right)/S_N \tag{71}$$

As a small sanity check we can compare the number of continuous degrees of freedom on both sides : on the left side we have N^2 degrees of freedom and the same number $N + (N^2 - N) = N^2$ we get on the right. Also let's mention that the permutation S_N corresponds to the case of N different eigenvalues, in other cases when two or more of them coincide one get's a submanifolds of zero Lebesgue measure which don't contribute to the matrix integrals.

Now let's notice that the integral over unitary group can be entirely factored out :

$$\mathcal{Z} = \int dU_{Haar} \int d\Lambda \Delta(\Lambda)^2 e^{-N\mathrm{tr}V(\Lambda)} \tag{72}$$

and looking at (71) we can immediately conclude :

$$\mathcal{Z} = \frac{\mathrm{vol}(U)}{(2\pi)^N N!} \int\limits_{\mathbb{R}^N} d\Lambda \Delta(\Lambda)^2 e^{-N\mathrm{tr}V(\Lambda)} \tag{73}$$

More accurately one could get this coefficient in the following way : let's choose quadratic potential $V(M) = M^2/2$, and perform the direct calculation on the r.h.s (it is just gaussian integrals) and compare with (23). Dividing the latter by the former and remembering about extra power of 2 from (69) one gets the constant $\int dU_{Haar}$ circumventing questions about explicit parametrization of the coset (71). Explicitly it reads as :

$$... \tag{74}$$

Comment on β-ensembles ....

<u>Normal Matrices</u> as analytic continuation of Hermitian Matrices.

Let's introduce Normal Matrices:

$$H_N(\{\gamma_i\}) \equiv \{M = U\Lambda U^{-1} \mid U \in U(N),\ \Lambda = diag\{\lambda_1, ..., \lambda_N\}, \lambda_i \in \gamma_i\} \tag{75}$$

where γ_i is an continuous curve in $\mathbb{C}$ without self-intersections. In particular case when $\gamma_i = \gamma = \mathbb{R}$ we get Hermitian matrices H_N while in the case of $\gamma_i = \gamma = S^1$ we get unitary matrices $U(N)$. In general case one can think about Normal Matrices as analytic continuation of Hermitian matrices wrt eigenvalues.

The corresponding partition function will have the following form :

$$\mathcal{Z} \sim \int\limits_{\prod_i \gamma_i} d\Lambda \Delta(\Lambda)^2 e^{-N \mathrm{tr} V(\Lambda)}. \tag{76}$$

The motivation for this generalization is at least two-folded, first even at the calculation of matrix integrals over Hermitian matrices one can find useful to deform the contour into the complex plane, for example in the context of saddle point approximation. Secondly for some potentials $V(M)$ the integral (73) simply doesn't exist however the proper choice of contours $\{\gamma_i\}$ can give a sense to the integral making it convergent.

Let's discuss the last point. Namely let's turn to the one-dimensional case and assume the cubic potential $V(\lambda) = \lambda^3 + c_2\lambda^2 + c_1\lambda^1 + c_0$. The density $e^{-V(\lambda)}$ is entire function and the convergence of $\int d\lambda e^{-V(\lambda)}$ does depend only on asymptotic behaviour on infinity. Particularly there are three allowed and three forbidden sectors for the asymptotic directions of the integration contour γ defined wrt to the behaviour of $e^{-\Re V(\lambda)}$, see Fig.8 and text below.

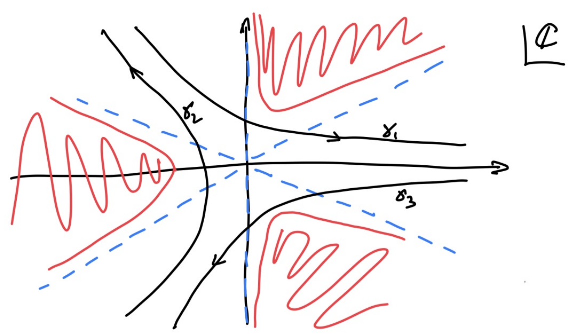

Figure 8: Red sectors (hills) are sectors of forbidden directions for $\int e^{-\lambda^3+\cdots} d\lambda$ with exponential growth of $e^{-\Re(\lambda^3+\ldots)}$. They are separated by blue punctured lines ($e^{i\pi/6}, e^{i(\pi/6+\pi/3)}, e^{i(\pi/6+2\pi/3)}, \ldots$) from allowed sectors (valleys) of exponential decrease of $e^{-\Re(\lambda^3+\ldots)}$. Contours $\{\gamma_i\}$ corresponding to convergent integral $\int d\lambda e^{-\lambda^3+\cdots}$ are going from infinity to infinity along the valleys. They are not independent because $\gamma_1 + \gamma_2 + \gamma_3 = 0$ is homotopic to zero, so one can choose two of them (up to homotopy) as a basis, e.g. $\{\gamma_1, \gamma_2\}$

Namely any contour giving convergent integral goes from infinity to infinity in the directions of the exponential decrease of $e^{-\Re(\lambda^3+\ldots)}$. Homotopic deformations of the contour doesn't change the holomorphic integral so all contours are defined up to homotopy. The last particularly imply that the sum $\gamma_1 + \gamma_2 + \gamma_3$ can be deformed to one point and thus there are only two independent contours. In the general case of polynomial potential of degree n there will be $n-1$ independent contours and any other convergent contour can be decomposed over them.

Now let's consider a potential which is a sum of an polynomial $P(\lambda)$ and an pole : $V(\lambda) = P(\lambda) + \frac{1}{(\lambda-\alpha)^m}$. It brings one more singularity at α and one has to analyse the behaviour of $e^{\Re(-V(\lambda))}$ in its vicinity. It will lead to m forbidden sectors and m valleys along which contours can reach the pole. Adding one more pole of order m brings $m+1$ new contours : m are (e.g.) connecting adjacent allowed sectors around the pole and one more goes from the new pole to one of existing singularities. Actually the picture will be the same for any rational potential which has one m-pole singularity at α and behaves as $P(\lambda)$ at infinity. See an example at Fig.9

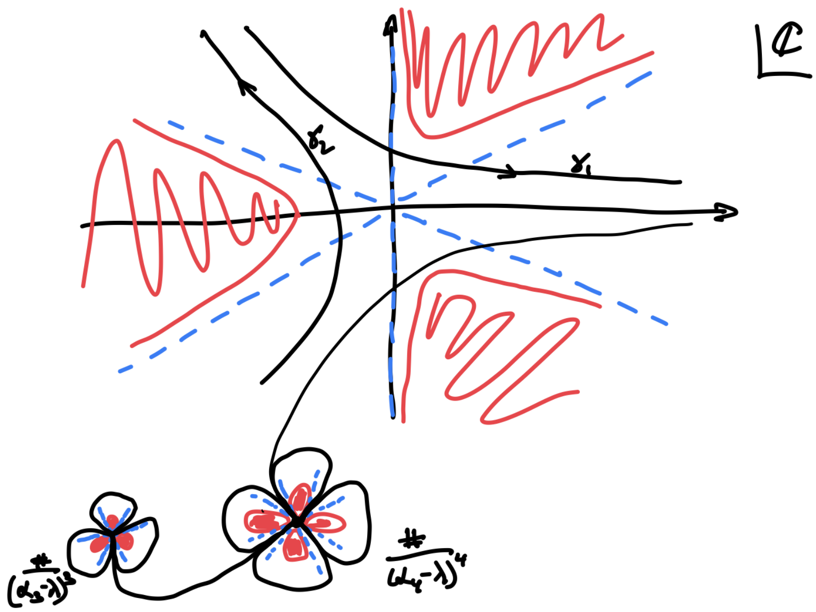

Figure 9: The allowed/forbidden areas structure for a rational potential behaving as λ^3 at infinity and having the poles of order 3 & 4 at points α_3 & α_4. The number of basis contours is equal to $11 = (3-1) + (3+1) + (4+1)$, they are colored black

2.2 Saddle point approximation

Here we briefly remind how the saddle-point approximation works.

2.3 Coulomb gas method and Spectral Curve

Omitting the prefactor we can rewrite the partition function (73) :

$$\mathcal{Z} = \int\limits_{\mathbb{R}^N} \prod d\lambda_i e^{-N^2 S(\lambda_1,...,\lambda_N)}, \tag{77}$$

$$S(\lambda_1, ..., \lambda_N) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{1}{N^2}\sum_{i<j}^{N} \log(\lambda_i - \lambda_j)^2 \tag{78}$$

Saddle point equation :

$$\frac{\partial S}{\partial \lambda_i} = 0 \ : \qquad V'(\lambda_i) = \frac{2}{N}\sum_{i<j}^{N} \frac{1}{\lambda_i - \lambda_j}, \qquad i \in \{1, ..., N\} \tag{79}$$

Now let's consider the real polynomial with the global minimum of $\Re(V(\lambda))$ on the real axe. The action $S(\{\lambda_i\})$ (78) can be interpreted as the energy of 2D electrons on the line at the background potential V and the S.P.Eqn is just the balance of forces telling us that the force induced by potential V is precisely compensated by the force acting from all other electrons. This simple intuitive picture can help us to imagine how the solution will look like. First let's introduce new auxillary parameter μ : $V'(\lambda_i) = \mu \frac{2}{N} \sum\limits_{i<j}^{N} \frac{1}{\lambda_i - \lambda_j}$. When $\mu = 0$ we have factorization of N-dimensional integral into the product of N one-dimensional integrals and all eigenvalues should be placed at the global minimum of $V(\lambda)$. When μ is finite but still small eigenvalues will start to experience the repulsion and the point corresponding to the minimum will blow into the finite interval. As bigger we make μ as bigger the interval will be. At some point it can be "energetically optimal" to have two or more separated intervals around different minima, see Fig.10

Going beyond the leading (exponential) order one can get contribution from other saddle points in the similar manner , namely at $\mu = 0$ one can describe all saddle points of partition function specifying the saddles of the one dimensional integral for every eigenvalue, in other words we can numerate all saddles and specify the solution by the collection of numbers $(N_1, N_2, ...)$, $\sum N_i = N$ telling us how many eigenvalues in the given saddle point. This numbers normalised by N are known as filling fractions $S_i = N_i/N$. At finite μ N_i eigenvalues around ith S.P. will blow into an interval (arc) but still filling fractions will be valid parameters to numerate saddle points.

Now, for sake of simplicity let's assume that we have finite number of intervals around one or several minima. Let's introduce density ρ

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - \lambda_i) \tag{80}$$

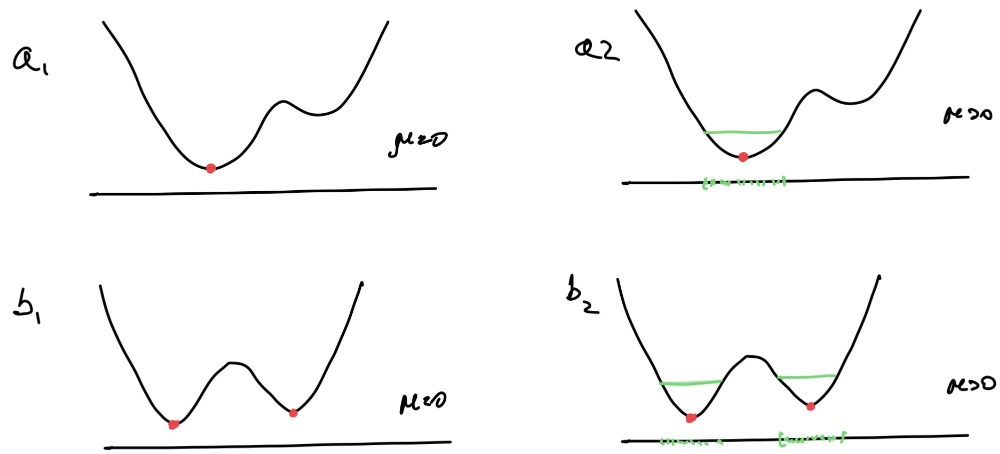

Figure 10: a1) : at $\mu = 0$ all eigenvalues seat in one global minimum a2) : at $\mu > 0$ eigenvalues are spreaded in an interval, b1) for more general saddle points at $\mu = 0$ eigenvalues can occupy several different saddles, b2) at $\mu > 0$ they all will be transformed into the intervals/curves

which we expect to have the support : $\text{supp}\rho = \cup(a_i, b_i)$ in the limit of large N.

Resolvent :

$$W(x) = \frac{1}{N} \sum \frac{1}{x - \lambda_i} \tag{81}$$

In the case of compact support it has the following asymptotic behaviour :

$$W(x) \underset{x\to\infty}{\longrightarrow} \frac{1}{x} \tag{82}$$

Using $\lim \frac{1}{x \pm i\epsilon} = p.v.\frac{1}{x} \mp i\pi\delta(x)$ we can restore the density :

$$\rho(x)|_{\text{supp}\rho} = \frac{1}{2\pi i}(W(x - i0)) - W(x + i0)) \tag{83}$$

and rewrite S.P.Eq. (79):

$$V'(x)|_{\text{supp}\rho} = W(x - i0)) + W(x + i0) \tag{84}$$

Using the definition (81) we can derive the following equation:

$$W^2(x) + \frac{1}{N}W'(x) = \frac{1}{N^2}\left(\sum_{i,j=1}^{N}\frac{1}{(x-\lambda_i)(x-\lambda_j)} - \sum_{i=1}^{N}\frac{1}{(x-\lambda_i)^2}\right) =$$

$$\frac{1}{N^2}\sum_{i\neq j}^{N}\frac{1}{(x-\lambda_i)(x-\lambda_j)} = \frac{2}{N^2}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j} = \frac{1}{N}\sum_{i=1}^{N}\frac{V'(\lambda_i)}{x-\lambda_i} \quad (85)$$

where we used $\frac{1}{(x-\lambda_i)(x-\lambda_j)} = \frac{1}{\lambda_i-\lambda_j}\left(\frac{1}{x-\lambda_i} - \frac{1}{x-\lambda_j}\right)$ and (79) in the last line. Introducing :

$$P(x) = \frac{1}{N}\sum_{i=1}^{N}\frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \quad (86)$$

we get :

$$W^2(x) + \frac{1}{N}W'(x) = V'(x)W(x) - P(x) \quad (87)$$

$P(x)$ is entire function (there is no poles at $\{\lambda_i\}$) with polynomial asymptotics at infinity and thus, by Liouville's theorem, it is a polynomial.

Let's notice that it is Riccati equation and can be linearised with the simple change of variables $W = \frac{1}{N}(\log\psi) = \frac{1}{N}\frac{\psi'}{\psi}$, $\psi = \prod_{i=1}^{N}(x-\lambda_i)$:

$$\frac{1}{N^2}\psi'' - \frac{1}{N}V'\psi' + P\psi = 0 \quad (88)$$

Comment One can notice the similarity with Jaynes-Cummings-Gauden model (a certain limit of Gauden model). Namely one can identify eigenvalues $\{\lambda_i\}$ with Bethe roots, $\psi(x)$ with Q-function and (88) with Baxter equation.

Now let's analyse the leading order in $1/N$. At this level we can skip second term in (87) , introduce : $\bar{W} = \lim_{N\to\infty} W$, $\bar{P} = \lim_{N\to\infty} P$ and write:

$$\bar{W}(x)^2 - V'(x)\bar{W}(x) + \bar{P}(x) = 0 \quad (89)$$

with solution :

$$\bar{W}(x) = \frac{1}{2}(V'(x) - \sqrt{V'^2 - 4\bar{P}}) \quad (90)$$

and thus , using (83):

$$\bar{\rho}(x)|_{\text{supp}} = \frac{1}{2\pi}\sqrt{4\bar{P} - V'^2} \quad (91)$$

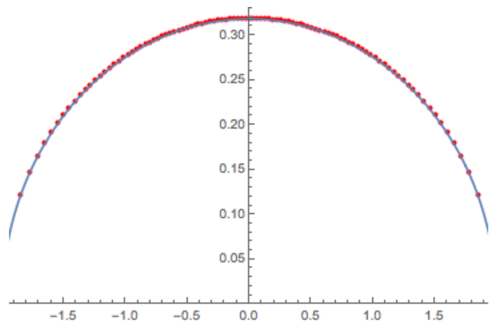

Figure 11: Red points : numerical solution of (79) for $V(\lambda) = \frac{1}{2}\lambda^2$, blue curve: Wigner's semicirlce (92)

Gaussian case : $V(\lambda) = \frac{1}{2}\lambda^2$: $\bar{P} = P = 1$ and we get famous Wigner's semicircle:

$$\bar{\rho}(x)|_{\text{supp}} = \frac{1}{2\pi}\sqrt{4 - x^2} \tag{92}$$

Let's check ourselve numerically : Fig.11

And compare with eigenvalue distribution of random matrix: Fig.12, Fig.13

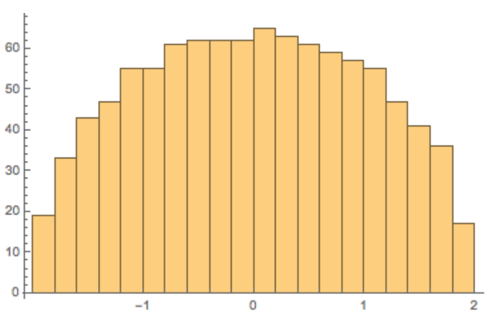

Figure 12: Histogram for eigenvalue distribution of 1000 × 1000 Hermitian ensemble with $V(\lambda) = \frac{1}{2}\lambda^2$

Let's also notice that the equation (88) can be exactly solved in this case by Hermite polynomials $\psi(x) = H_N(\sqrt{\frac{N}{2}}x)$. The asymptotics of the largest zero is $\approx \sqrt{2N}$ so we get that x is distributed on the interval $(-2, 2)$ in accordance with Wigner semicircle, see Fig.14

Now let's back to the general case. $V'^2 - 4\bar{P}$ is a polynomial which we can write as a product:

$$V'^2 - 4\bar{P} = M(x)^2\sigma(x) \tag{93}$$

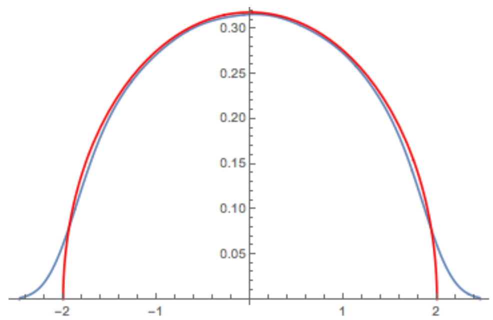

Figure 13: blue : smooth and normalized version of the same historgram; red : Wigner semicircle.

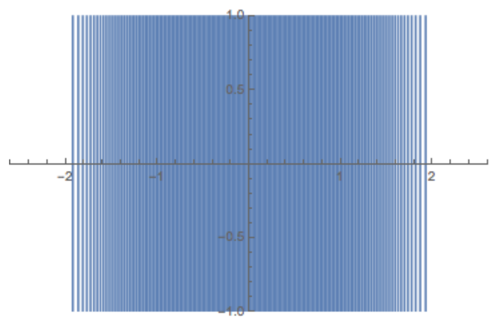

Figure 14: Distribution of zeros of Hermite Polynomial $H_{160}(\sqrt{\frac{160}{2}}x)$ of degree 160

of the polynomial $M(x)^2$ accumulating all even powers of zeros and $\sigma(x)$ - all single zeros. For example, we split $(x-a)(x-b)^2(x-c)^3 = M(x)^2\sigma(x)$ as $M(x)^2 = ((x-b)(x-c))^2$, $\sigma(x) = (x-a)(x-c)$. So we can rewrite resolvent and density :

$$\bar{W} = \frac{1}{2}(V' - M\sqrt{\sigma(x)}), \tag{94}$$

$$\bar{\rho}(x)|_{supp} = \frac{1}{2\pi}M(x)\sqrt{-\sigma(x)}. \tag{95}$$

Let's notice that $deg(V'^2 - 4\bar{P}) = deg(V'^2) = 2(d-1)$ and thus deg(σ) is even, so we can write :

$$\sigma(x) = \prod_{j=1}^{2s}(x - a_i) \tag{96}$$

Resolvent, actually can be written entirely through the polynomial $\sigma(x)$:

$$\frac{\bar{W}(x)}{\sqrt{\sigma(x)}} = \frac{1}{2\pi i}\oint_x \frac{dx'}{x'-x}\frac{\bar{W}(x')}{\sqrt{\sigma(x')}} = -\frac{1}{4\pi i}\oint_{supp}\frac{dx'}{x'-x}\left(\frac{V'(x')}{\sqrt{\sigma(x')}} - M(x')\right) \tag{97}$$

$$= -\frac{1}{4\pi i}\oint_{supp}\frac{dx'}{x'-x}\frac{V'(x')}{\sqrt{\sigma(x')}} = \frac{1}{2\pi i}\int_{supp}\frac{dx'}{x'-x}\frac{V'(x')}{\sqrt{\sigma(x')}} \tag{98}$$

where in the last line we used $\oint_{supp} dx'\frac{f(x')}{\sqrt{\sigma(x')}} = -2\int_{supp} dx'\frac{f(x')}{\sqrt{\sigma(x')}}$. So finally we get:

$$\bar{W}(x) = \frac{1}{2\pi i}\int_{supp} dx'\sqrt{\frac{\sigma(x)}{\sigma(x')}}\frac{V'(x')}{x'-x} \tag{99}$$

The resolvent $\bar{W}(x)$ is a solution of the algebraic equation (89):

$$y^2 - V'(x)y + \bar{P}(x) = 0 \tag{100}$$

and solution is a double-valued function (in (90) we choose minus sign). Instead one can think about it as a single-valued function defined on the compact Riemann surface Σ by the equation (100).

Namely we define the *spectral curve* $(\Sigma, x(z), y(z))$ as a Riemann surface Σ with immersion

$$\Sigma \hookrightarrow \bar{\mathbb{C}} \times \mathbb{C} : (x(z), y(z)) \in \bar{\mathbb{C}} \times \mathbb{C}, \quad z \in \Sigma \tag{101}$$

The Riemann surface Σ is compact due to the asymptotic behaviour (82) of the resolvent at infinity. All orientable compact Riemann surfaces are classified wrt their genus g and one can introduce the canonical basis $\{A_i, B_i\}, \quad i \in \{1..g\}$ of the first homology group $H_1(\Sigma)$ as depicted on Fig.15

The double-valued resolvent $\bar{W}(x)$ turns into the single-valued function $\bar{\omega}(z) = \bar{W}(x(z))$ on Σ. Particularly we can define the one-differential form $\bar{\omega}(z)dz$ and calculate its integral along A and B cycles.

A-periods give:

$$\frac{1}{2\pi i}\oint_{A_i}\bar{\omega}(z)dz = \int_{a_i}^{b_i}\bar{\rho}(x)dx = \frac{N_i}{N} = S_i \tag{102}$$

the fractions $S_i = N_i/N$ - number of eigenvalues at the interval (a_i, b_i).

In order to give interpretation to integrals over B-cycles let's introduce effective force $F_{eff} = V' - 2\int_{supp}\frac{\rho(x')}{x-x'}$ acting on an eigenvalue and its integral - effective potential :

$$V_{eff} = V - 2\int_{supp}\rho(x')\log|x-x'| + const \tag{103}$$

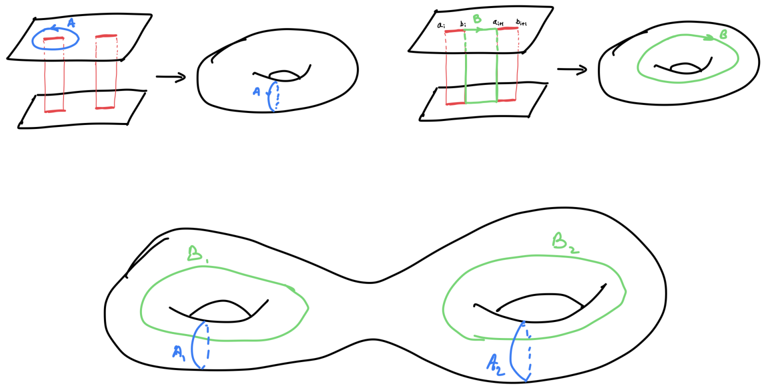

Figure 15: A and B-cycles. A_i goes around interval (a_i, b_i) while B_i from (a_i, b_i) to (a_{i+1}, b_{i+1}) and back.

It is equal to constant E_i on the interval (a_i, b_i) and changes somehow outside of support. Also let's notice that $V'_{eff} = V' - 2\bar{W} = M\sqrt{\sigma}$ has opposite sign on the second sheet, so we get:

$$\oint_{B_i} \bar{\omega}(z)dz = \frac{1}{2}\oint_{B_i} dz(V'(z) - V'_{eff}(z)) = -\int_{b_i}^{a_{i+1}} dzV'_{eff}(z) = E_i - E_{i+1} \quad (104)$$

what can be interpreted as a change of partitian function under the move of an eigenvalue from (a_{i+1}, b_{i+1}) to (a_i, b_i).

<u>One-cut solution</u>

Now let's discuss one-cut case in more details. First, let's introduce Joukowsky map :

$$x = \frac{a+b}{2} + \gamma(z + z^{-1}), \quad \gamma = \frac{b-a}{4} \quad (105)$$

and its inverse:

$$z = \frac{1}{2\gamma}\left(x - \frac{a+b}{2} \pm \sqrt{\left(x - \frac{a+b}{2}\right)^2 - 4\gamma^2}\right) \quad (106)$$

In z-variable we rewrite $\sqrt{\sigma(x)}$ as a rational function :

$$\sqrt{\sigma(x)} = \sqrt{(x-a)(x-b)} = \gamma(z - 1/z) \quad (107)$$

while in x plane we define square root such that $\sqrt{\sigma(x)} \sim x \sim \gamma z$ at $x \to \infty$ in the upper sheet and $\sqrt{\sigma(x)} \sim -x \sim -\gamma/z$ in the lower one.

Thus $\bar{W} = \frac{1}{2}(V' - M\sqrt{\sigma(x)})$ is also rational function in z and its leading asymptotic behaviour (see (81)) on the first sheet is $\bar{W}(x) \to 1/x$ so :

$$\bar{\omega}(z) = \sum_{k=0}^{d-1} v_k z^{-k} \tag{108}$$

$$v_0 = 0, \quad v_1 = 1/\gamma \tag{109}$$

Saddle Point Equation (84) written in z variable :

$$V'(x(z)) = \bar{W}(x(z) + i0) + \bar{W}(x(z) - i0) = \bar{\omega}(z) + \bar{\omega}(1/z) \tag{110}$$

or using (108) :

$$V'\left(\frac{a+b}{2} + \gamma(z + z^{-1})\right) = \sum_{k=0}^{d-1} v_k(z^k + z^{-k}) \tag{111}$$

what gives us d equations for d unknown parameters : $\{v_2, ..., v_{d-1}; a, b\}$.

Example Quadratic potential : $V(x) = \frac{x^2}{2}$

$V'(x) = x$ and SPE (111) reads as :

$$\frac{a+b}{2} + \gamma(z + z^{-1}) = \frac{1}{\gamma}(z + \frac{1}{z}) \tag{112}$$

what gives $a = -b = -2$, $\bar{\omega}(z) = z^{-1}$and we reproduce Wigner semicircle :

$$\bar{W}(x) = \frac{1}{2}(x - \sqrt{x^2 - 4}) \tag{113}$$

Now let's consider general symmetric potential $V(x) = \sum\limits_{m=1}^{k} \frac{g_{2m}}{2m} x^{2m}$ with real coefficients.

Due to Z_2-symmetry we get $a = -b$ and can write the resolvent :

$$\bar{W}(x) = \frac{1}{2}(V'(x) - M(x)\sqrt{x^2 - b^2}) \tag{114}$$

where :

$$M(x) = \sum_{m=1}^{k} g_{2m} \sum_{n=0}^{m-1} \binom{2n}{n} \left(\frac{b^2}{4}\right)^n x^{2m-2n-2} \tag{115}$$

and b is defined through the following equation :

$$\frac{1}{2}\sum_{m=1}^{k} g_{2m} \binom{2m}{m} \left(\frac{b^2}{4}\right)^n = 1 \tag{116}$$

Corresponding density : $\rho(x) = \frac{1}{2\pi} M(x)\sqrt{b^2 - x^2}$

In the case of quartic potential : $V(x) = \frac{x^2}{2} + \frac{g_4}{4} x^4$:

$$\rho(x) = \frac{1}{2\pi}(1 + \frac{g_4}{2} b^2 + g_4 x^2)\sqrt{b^2 - x^2} \tag{117}$$

$$3g_4 b^4 + 4b^2 - 16 = 0 \tag{118}$$

Free energy:

$$E^0(g_4) = \int_{-b}^{b} dx \rho(x)(\frac{x^2}{2} + \frac{g_4 x^4}{4}) - \int\int dx_1 dx_2 \rho(x_1)\rho(x_2) \log|x_1 - x_2| \tag{119}$$

$$E^0(g_4) - E^0(0) = \frac{1}{24}(\frac{b^2}{4} - 1)(9 - \frac{b^2}{4}) - \log\frac{b}{2} \tag{120}$$

Expansion at $g_4 \to 0$:

$$b^2 = \frac{2}{3g_4}\left((1 + 12g_4)^{\frac{1}{2}} - 1\right) \approx 1 - 3g_4 + 18g_4^2 - 135g_4^3 + ... \tag{121}$$

$$E^0(g_4) - E^0(0) \approx \frac{g_4}{2} - \frac{9}{8}g_4^2 + \frac{9}{8}g_4^3 - \frac{189}{8}g_4^4 + ... \tag{122}$$

Also one can calculate correlation functions:

$$\langle \mathrm{tr} M^{2p} \rangle = \int_{-b}^{b} dx \rho(x) x^2 p = \frac{(2p)!}{p!(p+2)!}\left(\frac{b}{2}\right)^{2p}(2p + 2 - \frac{p}{4}b^2) \tag{123}$$

Now let's put $g_4 = 1$ and keep g_2 as a free parameter: $V(x) = g_2\frac{x^2}{2} + \frac{x^4}{4}$.

The density corresponding to one-cut solution should be positive :

$$\rho(x) = \frac{1}{2\pi}\left(x^2 + g_2 + b^2/2\right)\sqrt{b^2 - x^2} \geq 0, \quad \forall x \in (-b, b) \tag{124}$$

what means that $g_2 + b^2/2 \geq 0$. Using expression for b: $b^2 = \frac{2}{3}(\sqrt{g_2^2 + 12} - g_2)$ we get $g_2 \geq -2$. What happens for $g_2 < -2$? Turns out that at such value of parameter there is no one-cut solution anymore and support splits into two disconnected intervals : $supp = (-a, -b) \cup (b, a)$:

$$\bar{W}(x) = \frac{1}{2}(x^3 + g_2 x - x\sqrt{(x^2 - a^2)(x^2 - b^2)}) \tag{125}$$

$$a^2 = 2 - g_2, \quad b^2 = -g_2 - 2 \tag{126}$$

3 Loop Equations

3.1 Loop Equations for correlation functions.

For convergent integrals we have obvious equality :

$$\sum_{i,j} \int dM \frac{\partial}{\partial M_{ij}} \left((M^k)_{ij} e^{-N \mathrm{tr} V(M)} \right) = 0 \tag{127}$$

Using

$$\frac{\partial}{\partial M_{ij}} (M^k)_{ij} = \sum_{l=0}^{k-1} (M^l)_{ii} (M^{k-l-1})_{jj}, \tag{128}$$

$$\frac{\partial}{\partial M_{ij}} e^{-N \mathrm{tr} V(M)} = V'(M)_{ji} e^{-N \mathrm{tr} V(M)} \tag{129}$$

we get the simplest loop equation :

$$\sum_{l=0}^{k-1} \langle \mathrm{tr} M^l \mathrm{tr} M^{k-l-1} \rangle = N \langle \mathrm{tr} M^k V'(M) \rangle \tag{130}$$

It is valid for finite N and provides the relation between different correlation functions.

Similarly, one can start with

$$\sum_{i,j} \int dM \frac{\partial}{\partial M_{ij}} \left((M^{\mu_1})_{ij} \mathrm{tr} M^{\mu_2} ... M^{\mu_n} e^{-N \mathrm{tr} V(M)} \right) = 0 \tag{131}$$

and get more general

Loop Equations for Correlation functions :

$$\sum_{l=0}^{\mu_1 - 1} \langle \mathrm{tr} M^l \mathrm{tr} M^{\mu_1 - l - 1} \prod_{i=2}^{n} \mathrm{tr} M^{\mu_i} \rangle + \sum_{j=2}^{n} \mu_j \langle \mathrm{tr} M^{\mu_1 + \mu_j - 1} \prod_{i=2, i \neq j}^{n} \mathrm{tr} M^{\mu_i} \rangle$$
$$= N \langle \mathrm{tr} V'(M) M^{\mu_1} \prod_{i=2}^{n} \mathrm{tr} M^{\mu_i} \rangle \tag{132}$$

The same equation and its generalisation to general β ensembles can be obtained from the eigenvalue representation:

$$\mathcal{Z} = \int_{\mathbb{R}^N} \Delta(\lambda)^\beta d\lambda_1 ... d\lambda_N e^{-N \frac{\beta}{2} \sum_{i=1}^{N} V(\lambda_i)}, \tag{133}$$

$$\frac{1}{\mathcal{Z}} \sum_{i=1}^{N} \int_{\mathbb{R}^N} d\lambda_1 ... d\lambda_N \frac{\partial}{\partial \lambda_i} \left(\Delta(\lambda)^\beta e^{-N \frac{\beta}{2} \sum_{i=1}^{N} V(\lambda_i)} \lambda_i^k \right) = \tag{134}$$

$$\sum_{i=1}^{N} \frac{\beta}{2} \langle \sum_{j \neq i} \frac{2 \lambda_i^k}{\lambda_i - \lambda_j} - N V'(\lambda_i) \lambda_i^k + \frac{2k}{\beta} \lambda_i^{k-1} \rangle = 0 \tag{135}$$

Using

$$\sum_{i=1}^{N}\sum_{j\neq i}\frac{2\lambda_i^k}{\lambda_i-\lambda_j}=\sum_{i=1}^{N}\sum_{j\neq i}\frac{\lambda_i^k-\lambda_j^k}{\lambda_i-\lambda_j}=\sum_{l=0}^{k-1}(\sum_i \lambda_i^l)(\sum_j \lambda_j^{k-l-1})-k\sum_i \lambda_i^{k-1} \tag{136}$$

We get :

$$\langle\sum_{l=0}^{k-1}(\sum_i \lambda_i^l)(\sum_j \lambda_j^{k-l-1})\rangle+(\frac{2}{\beta}-1)\langle k\sum_i \lambda_i^{k-1}\rangle=N\langle\sum_i V'(\lambda_i)\lambda_i^k\rangle \tag{137}$$

or rewriting it through M we get β version of (130) :

$$\sum_{l=0}^{k-1}\langle \mathrm{tr}M^l \mathrm{tr}M^{k-l-1}\rangle+(\frac{2}{\beta}-1)k\langle \mathrm{tr}M^{k-1}\rangle=N\langle \mathrm{tr}M^k V'(M)\rangle \tag{138}$$

Similarly one can get more general version:

$$\sum_{l=0}^{\mu_1-1}\langle \mathrm{tr}M^l \mathrm{tr}M^{\mu_1-l-1}\prod_{i=2}^{n}\mathrm{tr}M^{\mu_i}\rangle+\frac{2}{\beta}\sum_{j=2}^{n}\mu_j\langle \mathrm{tr}M^{\mu_1+\mu_j-1}\prod_{i=2,i\neq j}^{n}\mathrm{tr}M^{\mu_i}\rangle$$
$$+(\frac{2}{\beta}-1)\mu_1\langle \mathrm{tr}M^{\mu_1-1}\prod_{i=2}^{n}\mathrm{tr}M^{\mu_i}\rangle=N\langle \mathrm{tr}V'(M)M^{\mu_1}\prod_{i=2}^{n}\mathrm{tr}M^{\mu_i}\rangle \tag{139}$$

Comment on Loop Equations as Ward identities

Another way to think about loop equations is as a manifestation of the symmetry. In physics such relations are called as Ward identities or Swinger-Dyson equations. Namely let's notice that the matrix integral $\int dMe^{-N\mathrm{tr}V(M)}$ is invariant under the infinytisemal transformation $M\to M+\epsilon M^k$, then the change of variables gives the Jacobian :

$$\frac{d(M+\epsilon M^k)}{dM}=1+\epsilon\sum_{i,j}\frac{\partial}{\partial M_{ij}}(M^k)_{ij}+O(\epsilon^2)=1+\epsilon\sum_{l=0}^{k-1}\mathrm{tr}M^l\mathrm{tr}M^{k-l-1}+O(\epsilon^2) \tag{140}$$

while the variation of $e^{-N\mathrm{tr}V(M)}$ gives $\epsilon M^kV'(M)e^{-N\mathrm{tr}V(M)}$ and we get (130). More general change of variables gives further equations, particularly it is possible to introduce dif operators L_j , $j=-1,0,1,...$ and rewrite loop equations in the form of Virasoro - Witt algebra:

$$[L_k,L_j]=(k-j)L_{k+j} \tag{141}$$

We will return to this point later.

Comment on Loop Equations as Tutte's recursion

3.2 Loop Equations for resolvents and $1/N$ expansion. First few orders and the glimpse of Topological Recursion

Let's multiply (130) by x^{-k-1} and sum over k :

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k-1}\langle\frac{\mathrm{tr}M^l}{x^{l+1}}\frac{\mathrm{tr}M^{k-l-1}}{x^{k-l}}\rangle=\sum_{k=0}^{\infty}N\langle\frac{\mathrm{tr}M^k}{x^k}V'(M)\rangle \tag{142}$$

rewriting loop equation through the resolvents:

$$\langle\mathrm{tr}\frac{1}{x-M}\mathrm{tr}\frac{1}{x-M}\rangle-N\langle\mathrm{tr}\frac{V'(M)}{x-M}\rangle=0 \tag{143}$$

In what follows it will be useful to writ all equations through the connected resolvents (56):

$$W(x_1,...,x_n)=\langle\mathrm{tr}\frac{1}{x_1-M}...\mathrm{tr}\frac{1}{x_n-M}\rangle_c \tag{144}$$

while for ordinary expectation value we adopt hat notation $\hat{W}(x_1,...,x_n)$. Particularly for two-point resolvent we get from (56):

$$\hat{W}(x_1,x_2)=W(x_1,x_2)+W(x_1)W(x_2) \tag{145}$$

So we can rewrite (143):

$$W_2(x,x)+W_1^2(x)=N(V'(x)W_1(x)-P_0(x)) \tag{146}$$

where $P_0(x)=\langle\mathrm{tr}\frac{V'(x)-V'(M)}{x-M}\rangle$. We can also introduce more general polynomial (in x) :

$$P_n(x,x_1,...,x_n)=\langle\mathrm{tr}\frac{V'(x)-V'(M)}{x-M}\prod_{i=1}^{n}\mathrm{tr}\frac{1}{x_i-M}\rangle_c \tag{147}$$

and rewrite general loop equations (3.1) through the connected resolvents:

$$\begin{gathered}W_{n+2}(x,x,I)+\sum_{J\subset I}W_{1+|J|}(x,J)W_{1+n-|J|}(x,I\setminus J)+\\ \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{W_n(x,I\setminus\{x_i\}-W_n(I))}{x-x_i}=N(V'(x)W_{n+1}(x,I)-P_n(x,I))\end{gathered} \tag{148}$$

The derivation can be found in [2].

So far we assumed finite N, now let's turn to the limit of large N. From

(54) we get :

$$W_n = \sum_{g=0}^{\infty} N^{2-2g-n} W_{g,n}, \tag{149}$$

$$P_n = \sum_{g=0}^{\infty} N^{1-2g-n} P_{g,n}, \tag{150}$$

$$W_0 = F, \quad F = \sum_{g=0}^{\infty} N^{2-2g} F_g \tag{151}$$

Plugging this expansion into the first Loop Equation (143) and keeping only the first leading term we get :

$$W_{0,1}^2(x) = V'(x) W_{0,1}(x) - P_{0,0}(x) \tag{152}$$

which precisely coincides with the saddle point equation (89) we got in the previous chapter! It means that solution for $W_{0,1}$ can be described with spectral curve as it was in length discussed in the "Coulomb gas" chapter. In this chapter we will see how to find higher orders $W_{g,n}$ from loop equations using topological recursion, but before that let's make a quick overview of some basic results from the theory of hyperelliptic curves.

Summary of basic facts about hyperelliptic curves

The solution for $W_{0,1}$ is described by hyperelliptic curve :

$$y^2 = (V' - 2W_{1,0})^2 = M^2 \sigma \tag{153}$$

Square roots are defined as :

$$\text{1st (physical) sheet :} \quad x^{-s}\sqrt{\sigma(x)} \xrightarrow[x\to\infty]{} +1, \tag{154}$$

$$\text{2nd sheet :} \quad x^{-s}\sqrt{\sigma(x)} \xrightarrow[x\to\infty]{} -1 \tag{155}$$

where we remind $2s = deg(\sigma(x))$.
For any point x on the first sheet we note $\bar{x}$ the corresponding point on the second sheet :

$$y(\bar{x}) = -y(x), \quad M(\bar{x}) = M(x), \quad \sqrt{\sigma(\bar{x})} = -\sqrt{\sigma(x)}, \quad dx = d\bar{x} \tag{156}$$

and at branch points a_i:

$$\bar{a}_i = a_i, \quad i = 1, ..., 2s \tag{157}$$

In the vicinity of a_i we can define local coordinates $x = a_i + \tau_i^2$ such that $\tau_i(x) = \sqrt{x - a_i} = -\tau_i(\bar{x})$ and $dx = 2\tau_i d\tau_i$.

Let's notice that for any polynomial $L(x)$ with $deg(L) \leq s-2$ the differential form $L(x)\frac{dx}{\sqrt{\sigma(x)}}$ is regular on the whole surface and the following theorem has place:

Theorem

There are $s-1$ polynomials $L_j(x)$ of $deg = s-2$ such that : []

$$\oint_{[a_{2l-1},a_{2l}]} \frac{L_j(x)}{\sqrt{\sigma(x)}}dx = 2i\pi\delta_{l,j}, \quad \forall\ l,j \in [1,...,s-1] \tag{158}$$

$\{\frac{L_j}{\sqrt{\sigma}}\}$ are normalized holomorphic differentials and L_j's form a basis of $deg \leq s-2$ polynomials :

$$\forall P(x), \quad deg(P) \leq s-2 \ : \ P(x) = \sum_{j=1}^{s-1} \left(\frac{1}{2i\pi} \oint_{[a_{2j-1},a_{2j}]} \frac{P(x')}{\sqrt{\sigma(x')}}dx' \right) L_j(x) \tag{159}$$

Bergmann kernel (fundamental differential of the 2nd kind) is a unique meromorphic symmetric bilinear differential form $B(x_1,x_2)$ whose only singularity is a double pole at $x_1 - x_2$ such that:

$$\begin{cases} B(x_1,x_2) \underset{x_1\to x_2}{\sim} \frac{dx_1 dx_2}{(x_1-x_2)^2} + O(1) \\ \forall j = 1,...,s-1, \quad \oint_{x\in[a_{2j-1},a_{2j}]} B(x_1,x_2) = 0 \end{cases} \tag{160}$$

with the following explicit expression:

$$B(x_1,x_2) = \frac{1}{2}dx_2\frac{\partial}{\partial x_2}(\frac{dx_1}{x_1-x_2} + dS(x_1,x_2)), \tag{161}$$

$$dS(x_1,x_2) = \frac{\sqrt{\sigma(x_2)}}{\sqrt{\sigma(x_1)}} \left(\frac{1}{x_1-x_2} - \sum_{j=1}^{s-1} c_j(x_2)L_j(x_1) \right) dx_1, \tag{162}$$

$$c_j(x_2) = \frac{1}{2i\pi} \oint_{x\in[a_{2j-1},a_{2j}]} \frac{dx}{\sqrt{\sigma(x)}}\frac{1}{x-x_2} \tag{163}$$

where we also introduced $dS(x_1,x_2)$ - normalised differential of the 3rd kind.

Now let's describe the "boundary conditions" which will specify the solution of Loop Equations. For the potential $V(M) = \sum\limits_k \frac{g_k}{k} M^k$ let's introduce the following differential operator:

$$\frac{\partial}{\partial V(x)} := -\sum_k \frac{k}{x^{k+1}} \frac{\partial}{\partial g_k} \tag{164}$$

We have :

$$\frac{\partial}{\partial V(x)} \mathcal{Z} = \frac{\partial}{\partial V(x)} \int e^{-N\mathrm{tr}V(M)} = \langle \mathrm{tr} \frac{1}{x-M} \rangle = NW_1(x) \tag{165}$$

$$\frac{\partial}{\partial V(x_2)} W_1(x) = \frac{\partial}{\partial V(x_2)} \left(\frac{1}{\mathcal{Z}} \int \mathrm{tr} \frac{1}{x_1 - M} e^{-N\mathrm{tr}V(M)} \right) =$$
$$\frac{N}{\mathcal{Z}} \int \frac{1}{x_1 - M} \frac{1}{x_2 - M} e^{-N\mathrm{tr}V(M)} - \frac{N}{\mathcal{Z}^2} \int \frac{1}{x_1 - M} e^{-N\mathrm{tr}V(M)} \int \frac{1}{x_2 - M} e^{-N\mathrm{tr}V(M)}$$
$$= NW_2(x_1, x_2) \tag{166}$$

and in general :

$$W_{k+1}(x_1, ..., x_{k+1}) = \frac{\partial}{\partial V(x_{k+1})} W_k(x_1, ..., x_k) \tag{167}$$

We will specify the solution of loop equation fixing the fractions:

$$\frac{1}{2\pi i} \oint\limits_{[a_{2j-1}, a_{2j}]} W_1(x) dx = s_j = \lim_{N\to\infty} \frac{N_j}{N}, \quad \forall j = 1, ..., s \tag{168}$$

Acting by $\frac{\partial}{\partial V(y)}$ we get also:

$$\frac{1}{2\pi i} \oint\limits_{[a_{2j-1}, a_{2j}]} W_k(x_1, ..., x_k) dx_1 = 0, \quad \forall j = 1, ..., s, \quad k > 1 \tag{169}$$

Also we assume that there are no poles outside of the cuts:

$$\oint\limits_z W_k(x_1, ..., x_k) dx_1 = 0, \quad \forall z \in \mathbb{C} / \cup [a_{2j-1}, a_{2j}] \tag{170}$$

Conditions (168),(169),(170) were derived for formal matrix integrals, however we will assume them for convergent matrix integrals as well. Now let's get the leading order $W_{0,2}(x_1, x_2)$ of two-point resolvent. Second loop equation (148) for $n = 1$ reads as:

$$W_3(x, x, x_1) + 2W_1(x) W_2(x, x_1) + \frac{\partial}{\partial x_1} \frac{W_1(x) - W_1(x_1)}{x - x_1} = N(V'(x) W_2(x, x_1) - P_1(x, x_1)) \tag{171}$$

Leading order in $1/N$:

$$2W_{0,1}(x)W_{0,2}(x,x_1)+\frac{\partial}{\partial x_1}\frac{W_{0,1}(x)-W_{0,1}(x_1)}{x-x_1}=V'(x)W_{0,2}(x,x_1)-P_{0,1}(x,x_1) \tag{172}$$

Using $W_{0,1}=\frac{1}{2}(V'-M\sqrt{\sigma})$:

$$W_{0,2}(x,x_1)=-\frac{1}{2\sqrt{\sigma(x)}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{\sigma(x)}-\sqrt{\sigma(x_1)}}{x-x_1}\right)+\frac{R_2(x,x_1)}{M(x)\sqrt{\sigma(x)}} \tag{173}$$

where $R_2(x,x_1)$ is a polynomial of degree $\leq d-3$:

$$R_2(x,x_1)=\frac{1}{2}\frac{\partial}{\partial x_1}\frac{V'(x)-V'(x_1)}{x-x_1}-\frac{1}{2}\frac{\partial}{\partial x_1}\frac{M(x)-M(x_1)}{x-x_1}+P_{0,1}(x,x_1) \tag{174}$$

There should not be poles at zeros of $M(x)$ so $R_2(x,x_1)=M(x)\tilde{R}_2(x,x_1)$. In turn, $\tilde{R}_2(x,x_1)$ being of degree $\leq s-2$ can be expand over $\{L_j(x)\}$ (159)

$$\tilde{R}_2(x,x_1)=\sum_{j=1}^{s-1}\left(\frac{1}{2i\pi}\oint_{[a_{2j-1},a_{2j}]}\frac{\tilde{R}_2(x',x_1)}{\sqrt{\sigma(x')}}dx'\right)L_j(x) \tag{175}$$

using (173), $\frac{\tilde{R}_2(x',x_1)}{\sqrt{\sigma(x')}}$ can be written as:

$$\frac{\tilde{R}_2(x',x_1)}{\sqrt{\sigma(x')}}=W_{0,2}(x,x_1)+\frac{1}{2\sqrt{\sigma(x)}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{\sigma(x)}-\sqrt{\sigma(x_1)}}{x-x_1}\right) \tag{176}$$

$\oint_{[a_{2j-1},a_{2j}]} W_{0,2}(x,x_1)$ vanishes due to (169) and we get:

$$\begin{aligned}W_{0,2}&=-\frac{1}{2(x-x_1)^2}+\frac{1}{2}\frac{\partial}{\partial x_1}\frac{\sqrt{\sigma(x_1)}}{\sqrt{\sigma(x)}}\left(\frac{1}{x-x_1}-\sum_l c_l(x_1)L_j(x)\right)\\&=-\frac{1}{2(x-x_1)^2}+\frac{\partial}{\partial x_1}\frac{dS(x,x_1)}{dx}\end{aligned} \tag{177}$$

or through the Bergman kernel:

$$W_{0,2}(x_1,x_2)dx_1dx_2=B(x_1,x_2)-\frac{dx_1dx_2}{(x_1-x_2)^2} \tag{178}$$

Going further one can get $W_{1,1}$ from (146) in the next order:

$$2W_{0,1}(x)W_{1,1}(x)+W_{0,2}(x,x)=V'(x)W_{1,1}(x)-P_{1,0}(x), \tag{179}$$

$$W_{1,1}(x)=\frac{W_{0,2}(x,x)+P_{1,0}(x)}{M(x)\sqrt{\sigma(x)}} \tag{180}$$

At this point let's restrict our consideration to the

One-cut case In the one cut case we can use Joukowsky map $x(z) = \frac{a+b}{2} + \frac{a-b}{4}(z + z^{-1})$ and introduce the following set of differential forms:

$$\omega_{g,n}(z_1, ..., z_n) = W_{g,n}(x_1, ..., x_n)dx_1...dx_n + \delta_{g,0}\delta_{n,2}\frac{dx_1 dx_2}{(x_1 - x_2)^2} \tag{181}$$

Performing recursive analysis of the analytical structure of the loop equations (see [2] for detailes) one can show :

- $\omega_{g,n}(z_1, ..., z_n)$ is a rational symmetric function
- $\omega_{g,n}(\frac{1}{z_1}, ..., z_n) = -\omega_{g,n}(z_1, ..., z_n)$ for any (g, n) with negative euler characteristic $\chi(g, n) < 0$
- there are poles at $z_i = \pm 1$ only

We can write $W_{0,2}\&W_{1,1}$ as differential forms :

$$\omega_{0,2}(z, z') = B(z, z') = \frac{dz dz'}{(z - z')^2}, \tag{182}$$

$$\omega_{1,1}(z) = \frac{-B(z, z^{-1}) + P_{1,0}(x(z))d^2 x(z)}{-\omega(z)dx(z)} \tag{183}$$

where $\omega(z) = -M(x(z))\sqrt{\sigma(x(z))}$.

Using Cauchy's theorem, $\omega_{1,1}(z^{-1}) = -\omega_{1,1}(z)$ and the fact that its poles are only at ± 1 we get:

$$\begin{aligned} \omega_{1,1}(z) &= \frac{dz}{2\pi i}\oint_z \frac{1}{z' - z}\omega_{1,1}(z') = \\ &= \sum_{\pm}\frac{dz}{2\pi i}\oint_{\pm}\frac{1}{z - z'}\omega_{1,1}(z') = \\ &= \frac{1}{2}\sum_{\pm}\frac{dz}{2\pi i}\oint_{\pm}\left(\frac{1}{z - z'} - \frac{1}{z - 1/z'}\right)\omega_{1,1}(z') \end{aligned} \tag{184}$$

using (183) and the fact that $P_{1,0}(x(z))$ doesn't have poles at ± 1 we can further rewrite:

$$\omega_{1,1}(z) = \frac{1}{2}\sum_{\pm}\frac{dz}{2\pi i}\oint_{\pm}\left(\frac{1}{z - z'} - \frac{1}{z - 1/z'}\right)\frac{B(z', z'^{-1})}{\omega(z')dx(z')} \tag{185}$$

or introducing kernel K:

$$\omega_{1,1}(z) = \frac{1}{2}\sum_{\pm}\frac{dz}{2\pi i}\oint_{\pm} K(z, z')B(z', z'^{-1}), \tag{186}$$

$$K(z, z') = \frac{1}{2\omega(z')dx(z')}\left(\frac{1}{z - z'} - \frac{1}{z - 1/z'}\right) \tag{187}$$

Explicitly, evaluating residues :

$$\omega_{1,1}(z) = -\frac{1}{8(a-b)} \sum_{\eta=\pm 1} \frac{\eta}{\omega'(\eta)} \left(\frac{1}{(z-\eta)^4} + \frac{\eta}{(z-\eta)^3} - \frac{1 + \eta\frac{\omega''(\eta)}{\omega'(\eta)} + \frac{\omega'''(\eta)}{3\omega'(\eta)}}{2(z-\eta)^2} \right) dz \tag{188}$$

This representation can be generalised to higher orders $\omega_{g,n}$:

$$\omega_{g,n}(z_1, ..., z_n) = \sum_{\eta=\pm 1} \mathrm{Res}|_{z=\eta} K(z_1, z) \left[\omega_{g-1,n+1}(z, z^{-1}, z_2, ..., z_n) + \sum \omega_{h,1+|I|}(z, I)\omega_{h',1+|I'|}(z^{-1}, I') \right] \tag{189}$$

where sum goes over $h + h' = g$, $I \sqcup I' = \{z_2, ..., z_n\}$ excluding terms $(h, I) = (0, \emptyset)$ and $(h, I) = (g, \{z_2, ..., z_n\})$. This formula has recursive structure and expreses $\omega_{g,n}$ through the $\omega_{g',n'}$ with larger Euler characteristic $\chi(g', n') > \chi(g, n)$. It has generalisation to the multi-cut case and actually represents a particular example of much more general construction known as *Topological Recursion*.

3.3 Further comments on Topological Recursion

4 Orthogonal Polynomials

Partition function in the eigenvalue representation:

$$\mathcal{Z} = \frac{1}{N!} \int_{\mathbb{R}^N} d\lambda_0 ... d\lambda_{N-1} \Delta(\lambda_0, ..., \lambda_{N-1})^2 \prod_{i=0}^{N-1} e^{-NV(\lambda_i)} \tag{190}$$

using the determinant formula for Vandermonde :

$$\Delta(\lambda_0, ..., \lambda_{N-1}) = \det_{i,j}(\lambda_i^j) = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=0}^{N-1} \lambda_i^{\sigma(i)} \tag{191}$$

we can rewrite $\mathcal{Z}$ as a double-sum:

$$\mathcal{Z} = \frac{1}{N!} \sum_{\sigma,\sigma'} (-1)^\sigma (-1)^{\sigma'} \prod_{i=0}^{N-1} \int_{\mathbb{R}} d\lambda_i \lambda_i^{\sigma(i)} \lambda_i^{\sigma'(i)} e^{-NV(\lambda_i)} \tag{192}$$

Let's define the following scalar product:

$$\langle f|g \rangle = \int_{\mathbb{R}} d\lambda f(\lambda) g(\lambda) e^{-NV(\lambda)} \tag{193}$$

and rewrite :

$$\mathcal{Z} = \frac{1}{N!}\sum_{\sigma,\sigma'}(-1)^{\sigma}(-1)^{\sigma'}\prod_{i=0}^{N-1}\langle\lambda_i^{\sigma(i)}|\lambda_i^{\sigma'(i)}\rangle$$
$$= \sum_{\sigma}(-1)^{\sigma}\prod_{i=0}^{N-1}\langle\lambda_i^{\sigma(i)}|\lambda_i^{i}\rangle = \det_{i,j}\langle\lambda^i|\lambda^j\rangle \tag{194}$$

Let's notice that one can consider any polynomial $p_i = \lambda^i + \#\lambda^{i-1} + ...$ instead of monomials $\{\lambda^i\}$ and it will not change the determinant:

$$\mathcal{Z} = \det_{i,j}\langle p_i(\lambda)|p_j(\lambda)\rangle \tag{195}$$

Now let's choose the orthogonal basis of polynomials $\{p_i\}$: $\langle p_k|p_{k'}\rangle = h_k\delta$. The existence and uniqueness of such orthogonalisation follows from the application of Gram-Schmidt process (we assume that potential $V(\lambda)$ is real so the scalar product (193) is positive defined) to the monomials $\{\lambda^i\}$. In this basis the partition function has very simple form :

$$\mathcal{Z} = \prod_{k=0}^{N-1} h_k \tag{196}$$

Example 1 : $V(\lambda) = \frac{\lambda^2}{2}$ Orthogonal polynomials :

$$p_k(\lambda) = \frac{1}{(2N)^{k/2}}H_k(\lambda\sqrt{\frac{N}{2}}), \tag{197}$$

where H_k - Hermite polynomials : $H_k(x) = (-1)^k e^{\frac{x^2}{2}}\frac{d^k}{dx^k}e^{-\frac{x^2}{2}}$. Normalisation coefficients :

$$h_k = \frac{k!\sqrt{2\pi}}{N^{k+\frac{1}{2}}} \tag{198}$$

what gives:

$$\mathcal{Z} = \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N^2}{2}}}\prod_{k=0}^{N-1}k! = \frac{(2\pi)^{\frac{N}{2}}}{N^{\frac{N^2}{2}}}G(N+1) \tag{199}$$

where $G(x)$ is Barnes G-function.

Comment : With this result one can get the volume of the unitary group : $vol(U(N)) = \frac{(2\pi)^{\frac{N(N+1)}{2}}}{G(N+1)}$

Example 2 : Partition function of Chern-Simons Theory on S^3 and Stieltjes-Wigert matrix model

The partition function of the Chern-Simons theory on S^3 can be written as Stieltjes-Wigert matrix model [4]:

$$\mathcal{Z}_{SW} = \frac{1}{vol(U(N))} \int dM e^{-\frac{1}{2g_s} \text{tr}(\log M)^2} \tag{200}$$

Orthogonal Stieltjes-Wigert polynomials :

$$p_n(x) = (-1)^n q^{n^2+n/2} \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\frac{\nu(\nu-n)}{2} - \nu^2} (-q^{-\frac{1}{2}} x)^\nu \tag{201}$$

They are orthogonal wrt the measure $d\mu(x) = e^{-\frac{1}{2g_s} \log^2 x} dx$ and

$$h_n = \sqrt{2\pi g_s} q^{\frac{3}{4}n(n+1)+\frac{1}{2}} [n]! \tag{202}$$

where

$$q = e^{g_s}, \quad [n] = q^{n/2} - q^{-n/2}, \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!} \tag{203}$$

The Stieltjes-Wigert ensemble can be regarded as a q-deformation (in the sense of quantum group theory) of the usual Gaussian ensemble see [5, 4] for further details.

One of explicit expressions for orthogonal polynomials is given by **Heine's formula**

$$p_k(\lambda) = \langle \det(\lambda - M) \rangle_{k\times k} =$$
$$= \frac{1}{\mathcal{Z}_k} \int_{\mathbb{R}^k} d\lambda_0 ... d\lambda_{k-1} \Delta(\lambda_0, ..., \lambda_{k-1})^2 \prod_{i=0}^{k-1} (\lambda - \lambda_i) e^{-NV(\lambda_i)} \tag{204}$$

That's obviously the polynomial of degree k with coefficient 1 in the front of λ^k. Let's show that $\langle p_k | p_j \rangle = 0$, $\forall j < k$:

$$\mathcal{Z}_k \langle p_k | p_j \rangle = \int_{\mathbb{R}^{k+1}} d\lambda \left[d\lambda_0 ... d\lambda_{k-1} \Delta(\lambda_0, ..., \lambda_{k-1})^2 \prod_{i=0}^{k-1} (\lambda - \lambda_i) e^{-NV(\lambda_i)} \right] p_j(\lambda) e^{-NV(\lambda)} \tag{205}$$

Let's notice that :

$$\Delta(\lambda_0, ..., \lambda_{k-1}) \prod_{i=0}^{k-1} (\lambda - \lambda_i) = \Delta(\lambda_0, ..., \lambda_{k-1}, \lambda) \tag{206}$$

is tottaly antisymmetric over permutations of $(\lambda_0, ..., \lambda_{k-1}, \lambda)$ while antisymmetrization of $\Delta(\lambda_0, ..., \lambda_{k-1}) p_j(\lambda)$ gives zero ■

Exercise : Prove that antisymmetrization of $\Delta(\lambda_0, ..., \lambda_{k-1}) p_j(\lambda)$ gives zero.

4.1 Recursion relation

Multiplying $p_n(\lambda)$ by λ we get polynomial of degree $n+1$ which can be again decomposed over $\{p_1, ..., p_{n+1}\}$:

$$\lambda p_n(\lambda) = \sum_{k=0}^{n+1} a_k p_k(\lambda) \tag{207}$$

The leading coefficient $a_{n+1} = 1$ and $a_j = 0$ for $j < n-1$, what follows from $\langle \lambda p_n | p_k \rangle = \langle p_n | \lambda p_k \rangle = 0, \forall k < n-1$. So we can write the following:

<u>Recursion relation</u> :

$$\lambda p_n(\lambda) = p_{n+1}(\lambda) - s_n p_n(\lambda) + r_n p_{n-1}(\lambda) \tag{208}$$

Let's mention that for the even real potential $V(-\lambda) = V(\lambda)$ orthogonal polynomials have parity symmetry $p_n(-\lambda) = (-1)^n p_n(\lambda)$ (one can explicitly see it from the Heine's formula) and thus coefficients $s_n = 0$. Coefficient r_n can be expressed through h_n :

$$h_n = \langle p_n | \lambda p_{n-1} \rangle = \langle p_n \lambda | p_{n-1} \rangle = r_n h_{n-1} \tag{209}$$

so

$$r_n = \frac{h_n}{h_{n-1}} \tag{210}$$

And we can rewrite partition function:

$$\mathcal{Z} = \prod_{k=0}^{N-1} h_k = h_0^N \prod_{i=1}^{N-1} r_i^{N-i} \tag{211}$$

And Free Energy (by defenition):

$$\mathcal{F}(\mathbf{g}) = -\frac{1}{N^2} \log \frac{\mathcal{Z}(\mathbf{g})}{\mathcal{Z}(\mathbf{0})} = \\ = -\frac{1}{N} \log \frac{h_0(\mathbf{g})}{h_0(\mathbf{0})} - \frac{1}{N} \sum_{i=1}^{N-1} (1 - \frac{i}{N}) \log \frac{r_i(\mathbf{g})}{r_i(\mathbf{0})} \tag{212}$$

where $\mathbf{g} = \{g_i\}$ - collection of coupling constants : $V(\lambda) = \frac{1}{2}\lambda^2 - \sum\limits_{k=3}^{d} \frac{g_k}{k}\lambda^k$. So zero coupling constants correspond to Gaussian ensemble :

$$h_0(\mathbf{0}) = \sqrt{\frac{2\pi}{N}}, \qquad r_i(\mathbf{0}) = \frac{i}{N} \tag{213}$$

Now let's derive the equation for r_n :

$$nh_n = \langle p_n'\lambda|p_n\rangle = \langle p_n'|\lambda p_n\rangle = \langle p_n'|p_{n+1} - s_n p_n + r_n p_{n-1}\rangle =$$
$$= r_n\langle p_n'|p_{n-1}\rangle = r_n \int\limits_{\mathbb{R}} d\lambda e^{-NV(\lambda)} \left((p_n p_{n-1})' - p_n p_{n-1}'\right) = r_n \int\limits_{\mathbb{R}} d\lambda e^{-NV(\lambda)} (p_n p_{n-1})'$$
$$= N r_n \int\limits_{\mathbb{R}} d\lambda e^{-NV(\lambda)} V'(\lambda) p_n p_{n-1} \tag{214}$$

where we used the recursion relation in the second line and the integration by parts in the third. So we get "string equation" :

$$nh_n = N r_n \int\limits_{\mathbb{R}} d\lambda e^{-NV(\lambda)} V'(\lambda) p_n p_{n-1} \tag{215}$$

Now let's specialise our discussion to the quartic case : $V(\lambda) = \frac{\lambda^2}{2} + g\lambda^4$. In this case $V'(\lambda) = \lambda + 4g\lambda^3$ so we need to calculate $\langle p_n \lambda^3 p_{n-1}\rangle$. That's can be done just applying recursion relation several times to $\lambda^3 p_{n-1}$:

$$\lambda^3 p_{n-1} = \lambda^2(p_n + r_{n-1}p_{n-2}) = \lambda(p_{n+1} + r_n p_{n-1} + r_{n-1}(p_{n-1} + r_{n-2}p_{n-3})) \tag{216}$$

what gives us :

$$\langle p_n \lambda^3 p_{n-1}\rangle = h_n(r_{n+1} + r_n + r_{n-1}) \tag{217}$$

and the string equation (215) reads :

$$\frac{n}{N} = r_n(g)(1 + 4g(r_{n+1}(g) + r_n(g) + r_{n-1}(g))) \tag{218}$$

4.2 Large N and Double Scaling limit

In this section we closely follow

Let's introduce new variables sutable for the large N analysis :

$$\epsilon = \frac{1}{N}, \qquad x = \frac{n}{N}, \qquad r(x,g) = r_n(g) \tag{219}$$

and rewrite (218) :

$$x = r(x,g)\left(1 + 4g(r(x+\epsilon,g) + r(x,g) + r(x-\epsilon,g))\right) \tag{220}$$

Now let's **assume** the following ansatz :

$$r(x,g) = r_0(x,g) + \epsilon^2 r_2(x,g) + \epsilon^4 r_4(x,g) + ... \tag{221}$$

So we can write :

$$r(x+\epsilon,g)+r(x-\epsilon,g)=2\sum_{n=0}^{\infty}\epsilon^{2n}\sum_{k+p=n}\frac{1}{(2p)!}\frac{d^{2p}}{dx^{2p}}r_{2k}(x,g) \tag{222}$$

and plug into the (220) :

$$x\delta_{s,0}=r_{2s}(x,g)+4g\sum_{m+n=s}r_{2m}(x,g)\left(r_{2n}(x,g)+2\sum_{k+p=n}\frac{1}{(2p)!}\frac{d^{2p}}{dx^{2p}}r_{2k}(x,g)\right) \tag{223}$$

This formula has a recursive structure and r_{2s} can be calculated one by one, particularly for $s=0$ and $s=1$ we get :

$$r_0(x,g)=\frac{-1+\sqrt{1+48gx}}{24g}, \qquad r_2(x,g)=\frac{96g^2r_0(x,g)}{(1+48gx)^2} \tag{224}$$

In order to calculate the free energy (212) we can use Euler-Maclaurin formula :

$$\frac{1}{N}\sum_{n=1}^{N}f(\frac{n}{N})=\int_0^1 dsf(x)+\frac{1}{2N}f(x)\Big|_0^1+\sum_{n=1}^{p-1}\frac{B_{2n}}{(2n)!}\frac{1}{N^{2n}}f^{(2n-1)}(x)\Big|_0^1+O(\frac{1}{N^{2p+1}}) \tag{225}$$

where B_{2n} are Bernoulli numbers and p indicates the order of the approximation.

Choosing $f(x)=(1-x)\log\frac{r(x,g)}{x}$ we can use this formula to perform the sum into the (212) (mind the (213)) :

$$\begin{aligned}\mathcal{F}(g)=&-\frac{1}{N}\log\frac{h_0(g)}{h_0(0)}-\int_0^1 dx(1-x)\log\frac{r(x,g)}{x}+\frac{1}{2N}\lim_{x\to 0}\log\frac{r(x,g)}{x}\\&-\frac{1}{12N^2}\left((1-x)\log\frac{r(x,g)}{x}\right)'\Big|_0^1+O(\frac{1}{N^4})\end{aligned} \tag{226}$$

The coefficient $h_0(g)$ can be directly calculated :

$$h_0(g)=\langle 1|1\rangle=\int_{-\infty}^{\infty}d\lambda e^{-N(\lambda^2/2+g\lambda^4)}=\frac{e^{\frac{N}{32g}}}{2\sqrt{2g}}K_{\frac{1}{4}}(\frac{N}{32g}) \tag{227}$$

with regular $1/N$ expansion :

$$h_0(g)\sim\sqrt{\frac{2\pi}{N}}\left(1-\frac{3g}{N}+\frac{105g^2}{N^2}+...\right) \tag{228}$$

So we get :

$$\frac{1}{N^2}\mathcal{F} = -\int_0^1 dx(1-x)\log\frac{r(x,g)}{x}$$

$$-\frac{1}{N^2}\left[\int_0^1 dx(1-x)\frac{r_2(x,g)}{r_0(x,g)} + \frac{1}{12}\left((1-x)\log\frac{r_0(x,g)}{x}\right)'\Bigg|_0^1 - 3g\right] + O(\frac{1}{N^4}) \tag{229}$$

As was discussed in the first section , the Free energy has the following topological expansion (37):

$$\mathcal{F}(g) = \sum_{n=0}^{\infty}\mathcal{F}^h(g)N^{\chi(h)} \tag{230}$$

The direct calculation (as presented above) gives $\mathcal{F}^h$ owning the same critical point $g_c = -\frac{1}{48}$:

$$\mathcal{F}^0(g) \underset{g\to g_c}{=} \#(g-g_c)^{\frac{5}{2}} \tag{231}$$

$$\mathcal{F}^1(g) \underset{g\to g_c}{=} \#\log(g-g_c) \tag{232}$$

$$\mathcal{F}^h(g) \underset{g\to g_c}{=} \#(g-g_c)^{\frac{5}{4}\chi(h)} \tag{233}$$

Being combined with topological expansion it gives us the following sum of the most singular contributions :

$$\mathcal{F}(g) \to \sum \mathcal{F}^h(g)N^{\chi(h)} \to \sum f_n\left((g-g_c)^{\frac{5}{4}}N\right)^{\chi(h)} \tag{234}$$

what is calling for the double scaling limit : $N\to\infty$, $g-g_c\to 0$, $N(g-g_c)^{\frac{5}{4}} = fixed.$

Small comment on the intuition behind DS. As we will see in a minute such double scaling indeed exists, but before that let's intuitively try to understand what it should describe. Let's try to estimate the number of squares $\langle n\rangle$ in the typical graph of the given topology thinking about $\mathcal{F}^h(g) = \sum a_n g^n$ as a probability distribution : $\langle n\rangle = \frac{\sum n a_n g^n}{\sum a_n g^n} = \frac{g(\mathcal{F}^h(g))'}{\mathcal{F}^h(g)} \underset{g\to g_c}{\to} \frac{1}{g-g_c}$ so the number of vertices in the typical graph is going to infinity or in other words (making the appropriate rescaling of the square's area) we can see it as a very dense/smooth approximation of 2d surface. On the other hand in the DS we ressum the whole $1/N$ series so it's tempting to expect that this DS limit will describe 2d gravity and indeed it turns out to be the case. We will discuss it later in the one of the next sections, for the moment we will just proceed with the analysis of the matrix model.

In order to understand where the main contribution in the DS limit comes from let's write few more rs:

$$r_4 = \frac{\#}{(1+48gx)^5}, \quad r_6 = \frac{\#}{(1+48gx)^{\frac{15}{2}}}, \quad r_8 = \frac{\#}{(1+48gx)^10}, ... \tag{235}$$

so we see that at $g \to g_c = -1/48$ the most singular contribution comes from $x \to 1$. All these motivates the

DS limit :

$$N \to \infty, \quad g - g_c \to 0, \quad \kappa^{-1} = N(g-g_c)^{\frac{5}{4}} = fixed \tag{236}$$

and the following change of variables :

$$x = 1 - (g-g_c)z, \quad r(x,g) = r_0(1,g_c) + (g-g_c)^{\frac{1}{2}}\rho(z) \tag{237}$$

Plugging it into the (220) and taking the limit we get DS-equation which ghas the form of Painleve I:

$$\frac{\kappa^2}{6}\rho(z)'' + \frac{1}{4}\rho(z)^2 = z \tag{238}$$

This equation has the following solution in the form of power series:

$$\rho_{pert}(z) = z^{\frac{1}{2}} \sum_{n=0} a_n \left(\frac{\kappa}{z^{\frac{5}{4}}}\right)^{2n} \tag{239}$$

where coefficients satisfy recursion relation :

$$a_{n+1} = \frac{25n^2-1}{24}a_n + \frac{1}{4}\sum_{m=1}^{n} a_m a_{n+1-m}, \quad a_0 = -2, \quad n \geq 0 \tag{240}$$

with the following double-factorial asymptotics : $a_n \sim \left(\frac{5}{4\sqrt{6}}\right)^{2n-1/2}\Gamma(2n-1/2)$ what mean that the series is asymptotic and there are non-analytic terms. In order to find them, let's write $\rho(z) = \rho_{pert}(z) + \epsilon(z)$ and get the linearized equation for $\epsilon(z) \ll \rho_{pert}(z)$:

$$\frac{\kappa^2}{6}\epsilon''(z) + \frac{1}{2}\rho_{pert}(z) = 0 \tag{241}$$

Using WKB one gets :

$$\epsilon(z) = c\left(\frac{\kappa^{\frac{4}{5}}}{z}\right)^{\frac{1}{8}} e^{-\frac{4\sqrt{6}}{5\kappa}z^{\frac{5}{4}}} \left(1 + \#\left(\frac{\kappa^{\frac{4}{5}}}{z}\right) + \#\left(\frac{\kappa^{\frac{4}{5}}}{z}\right)^2 + ...\right) \tag{242}$$

4.2.1 Exponentially small terms and instantons

In order to understand the origin of this exponentially small term on the level of Matrix integral, let's consider the partition function separating explicitly the integration over one eigenvalue :

$$\mathcal{Z} = \int_{\mathbb{R}} d\lambda e^{-NV(\lambda)} \int_{\mathbb{R}^{N-1}} \prod_{i=1}^{N-1} d\mu_i \Delta_{N-1}(\mu_1, ..., \mu_{N-1})^2 e^{-N \sum_{i=1}^{N-1} V(\mu_i)} \prod_{i=1}^{N-1} (\lambda - \mu_i)^2$$
$$= \int_{\mathbb{R}} d\lambda e^{-NV_{eff}(\lambda)} \qquad (243)$$

and the effective potential can be written as :

$$V_{eff}(\lambda) = V(\lambda) - \int_{-a}^{a} d\mu \rho(\mu) \log(\lambda - \mu)^2 \qquad (244)$$

where we assume one-cut $(-a, a)$ solution. Let's mention that $V_{eff} = const$ on the interval $(-a, a)$ due to the equation of motion (79) : $V'_{eff} = 0$, so its value can be calculated e.g. at zero :

$$V_{eff}(0) = V(0) - \int_{-a}^{a} d\mu \rho(\mu) \log \mu^2 \qquad (245)$$

and we can rewrite the partition function as :

$$\mathcal{Z} = \int_{\mathbb{R}} d\lambda e^{-NV_{eff}(\lambda)} = 2ae^{-NV_{eff}(0)} \left(1 + \frac{1}{2a} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) d\lambda e^{-N(V_{eff}(\lambda) - V_{eff}(0))} \right) \qquad (246)$$

The integral outside the interval $(-a, a)$ can be calculated by saddle point approximation and the saddle point is given by the same equation : $V'_{eff}(\lambda^*) = 0$. In the case of quartic potential $g \in (g_c, 0)$ one gets :

$$\lambda_{\pm} = \pm \sqrt{-\frac{1}{6g} - \frac{\sqrt{1+48g}}{12g}}, \qquad (247)$$

$$V_{eff}(\lambda_{\pm}) - V_{eff}(0) = \sqrt{3} \frac{\sqrt{2+\sqrt{y}}}{1-y} y^{\frac{1}{4}} - 2\text{arctanh} \frac{\sqrt{2+\sqrt{y}}}{\sqrt{3} y^{\frac{1}{4}}} \qquad (248)$$

where $y = 1 - g/g_c$ and in the limit $y \to 0$:

$$V_{eff}(\lambda_{\pm}) - V_{eff}(0) = \frac{4}{5}\sqrt{6} y^{\frac{5}{4}} + \frac{2}{7}\sqrt{\frac{3}{2}} y^{\frac{7}{4}} + ... \qquad (249)$$

$$N(V_{eff}(\lambda_{\pm}) - V_{eff}(0)) = \frac{4}{5}\sqrt{6}(y^{\frac{5}{4}} N) \qquad (250)$$

The term surviving in the DS limit is the contribution streaming from exponential correction (242). So the origin of the exponentially small corrections is a "tunneling" of an eigenvalue from the cut to the other extremum. In principle one can move several eigenvalues and get next sublidding corrections. The spectral curve correspondiing to such "instanton" configuration is drawn at Fig.16 :

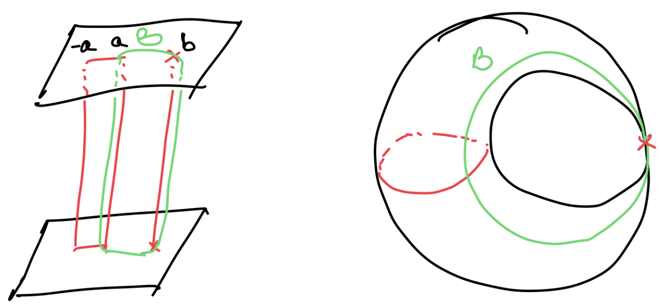

Figure 16: Spectral curve with the singular point corresponding to the tunneled eigenvalue

And the value of the exponentially small correction is given by B-cyrcle :

$$\oint_B \bar{\omega}(z)dz = \frac{1}{2}\oint_B (V'(z) - V'_{eff}(z))dz = -\int_a^b V'_{eff} = V_{eff}(a) - V_{eff}(b) \quad (251)$$

Further comments on Painleve I

After a proper rescaling, the DS equation (238) can be brought to the standard Painleve I form :

$$-\frac{1}{6}u'' + u^2 = z \quad (252)$$

One-parametric solution has the following form :

$$u(z,c) = \sum_{k\geq 0} c^k u_k(z), \quad (253)$$

$$u_k(z) = z^{\frac{1}{2}} e^{-kAz^{\frac{5}{4}}} \phi_k(z), \qquad \phi_k(z) = z^{-\frac{5k}{8}} \sum_{n\geq 0} u_{n,k} z^{-\frac{5n}{4}}, \qquad A = \frac{8\sqrt{3}}{5} \quad (254)$$

Asymptotics of $u_{n,k}$ know about each other, e.g. :

$$u_{n,0} \underset{n\to\infty}{\sim} A^{-2n+\frac{1}{2}}\Gamma(2n-\frac{1}{2})\frac{S_1}{i\pi}(1+\sum_{l=1}^{\infty}\frac{u_{l,1}A^l}{\prod_{k=1}^{l}(2n-\frac{1}{2}-k)}) \tag{255}$$

where $S_1 = -i\frac{3^{\frac{1}{4}}}{2\sqrt{\pi}}$ is a Stokes constant, see [6] for further details.

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