SUSY QM

1D $\qquad \varphi$ - e.f. $\quad \psi$ - Gr. var.

$$H = -\partial_x^2 + V(x)$$

$$H\varphi_0 = 0 \qquad \varphi_0 \text{ - ground state}$$

$$V(x) = \frac{\varphi_0''}{\varphi_0}$$

$$\overbrace{(-\partial_x + W')}^{A^+}\overbrace{(\partial_x + W')}^{A} = H$$

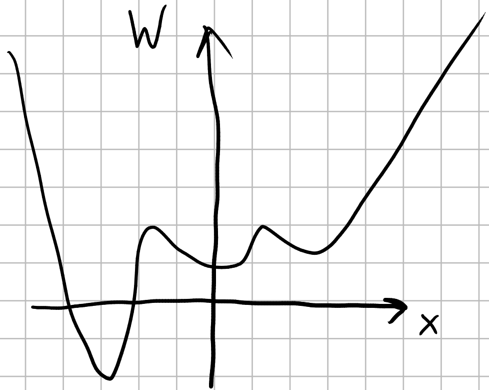
$$-\partial_x^2 + W'^2 + W'\partial_x \underbrace{-\partial_x W'}_{(\partial_x W') + W'\partial_x} = -\partial_x^2 + W'^2 - W'' = H$$

$$W'^2 - W'' = V \qquad\qquad H = A^+ A$$

$$A\varphi_0 = 0 = \varphi_0' + W'\varphi_0 \qquad\qquad W' = -\frac{\varphi_0'}{\varphi_0} = -(\ln\varphi_0)'$$

$$\varphi_0(x) = c\exp(-W(x))$$

$$\exists\,\varphi_0 \Longleftarrow \quad W(x) \begin{cases} \xrightarrow{x\to-\infty} +\infty \\ \xrightarrow{x\to} +\infty \end{cases}$$



Partner H.: $\quad H_2 = AA^+ = -\partial_x^2 + W'^2 + W''$

old $H \mapsto H_1$

$$H_1\varphi_n^{(1)} = \overset{\nearrow 0}{E_n^{(1)}}\varphi_n^{(1)} = A^+A\varphi_n^{(1)} \qquad \Big|\, A\times$$

$$H_2 A\varphi_n^{(1)} = \underbrace{AA^+}A\varphi_n^{(1)} = E_n^{(1)} A\varphi_n^{(1)}$$

$$H_2 \varphi_n^{(2)} = E_n^{(2)} \varphi_n^{(2)} = A A^+ \varphi_n^{(2)} \qquad \Big| A^+ \cdot$$

$$H_1 A^+ \varphi_n^{(1)} = A^+ A A^+ \varphi_n^{(2)} = E_n^{(1)} A^+ \varphi_n^{(2)}$$

$$Q_+ , Q_- , H \qquad [Q_+, H] = [Q_-, H] = 0$$

$$\{Q_+, Q_-\} = H$$

add (odd) $\uparrow$ Q_+, Q_-; even $\uparrow$ H

$$Q_+ = \psi \cdot A$$
$$Q_- = \partial_\psi \cdot A^+$$

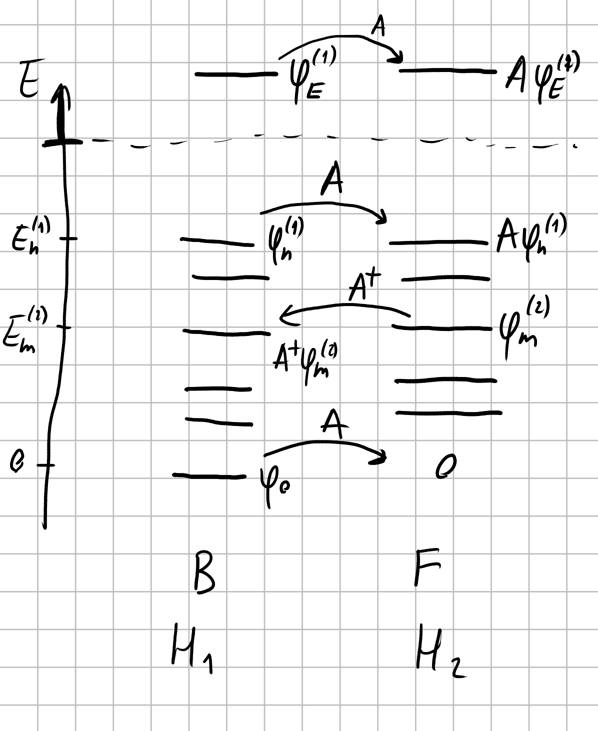
$$\{Q_+, Q_-\} = \{\psi A, \partial_\psi A^+\} = \underbrace{\psi \partial_\psi}_{\Pi_F} \underbrace{A A^+}_{H_2} + \underbrace{\partial_\psi \psi}_{\Pi_B} \underbrace{A^+ A}_{H_1} = H$$

$$\mathcal{H} = L^2(\mathbb{R}, \psi) = L^2(\mathbb{R}) + \psi L^2(\mathbb{R})$$

Π_B, Π_F

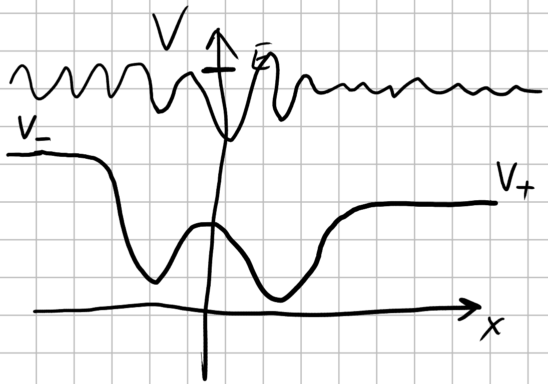
$$H \varphi_n = H(\varphi_n^B + \psi \varphi_n^F) = E_n (\varphi_n^B + \psi \varphi_n^F)$$

$$H Q_+ \varphi_n = (H_1 \Pi_B + H_2 \Pi_F)(\psi A \varphi_n^B) = H_2 \psi A \varphi_n^B \overset{(1)}{=} E_n \psi A \varphi_n^B = E_n Q_+ \varphi_n^B = E_n Q_+ \varphi_n$$



$$H^{(1)} \varphi_E^{(1)} = E^{(1)} \varphi_E^{(1)} \qquad \Big| A \cdot$$
$$H^{(1)} = A^+ A$$

$$A A^+ A \varphi_E^{(1)} = H^{(2)} A \varphi_E^{(1)} = E^{(1)} A \varphi_E^{(1)}$$



$$W'^2 - W'' \xrightarrow{-\infty} V_-$$
$$W'^2 - W'' \xrightarrow{+\infty} V_+$$

$$W' \xrightarrow{-\infty} \pm\sqrt{V_-} = W'_- = const \qquad p_+^2 = E^{(1)} - V_+$$
$$W' \xrightarrow{+\infty} \pm\sqrt{V_+} = W'_+ = const \qquad p_-^2 = E^{(1)} - V_-$$

$$\Phi_E^{(1)} : \quad H^{(1)}\Phi_E^{(1)} = E^{(1)}\Phi_E^{(1)} \qquad \Phi_E^{(1)} \xrightarrow{-\infty} e^{ip_-x} + R_E^{(1)} e^{-ip_-x}$$
$$\Phi_E^{(1)} \xrightarrow{+\infty} T_E^{(1)} e^{ip_+x}$$

$$A\Phi_E^{(1)} = (\partial_x + W')\Phi_E^{(1)} \xrightarrow{-\infty} (ip_- + W'_-)e^{ip_-x} + (-ip_- + W'_-)R_E^{(1)} e^{-ip_-x}$$
$$A\Phi_E^{(1)} \xrightarrow{+\infty} (ip_+ + W'_+)T_E^{(1)} e^{ip_+x}$$

$$\Phi_E^{(2)} = \frac{A\Phi_E^{(1)}}{ip_- + W'_-} \xrightarrow{-\infty} e^{ip_-x} + \underbrace{\frac{-ip_- + W'_-}{ip_- + W'_-} R_E^{(1)}}\, e^{-ip_-x}$$
$$\Phi_E^{(2)} \xrightarrow{+\infty} \underbrace{\frac{ip_+ + W'_+}{ip_- + W'_-} T_E^{(1)}}\, e^{ip_+x} \qquad R_E^{(2)} = e^{iS_R} R_E^{(1)}$$

$$T_E^{(2)} = e^{iS_T} T_E^{(1)} \qquad |R|^2 + |T|^2 = 1$$

Ex: $W'(x) = \ell \cdot th(x)$

$$H_1 = -\partial_x^2 + \ell^2 \underbrace{th^2 x}_{1 - \frac{1}{ch^2 x}} - \ell\frac{1}{ch^2 x} = -\partial_x^2 + \ell^2 - \frac{\ell(\ell+1)}{ch^2 x}$$

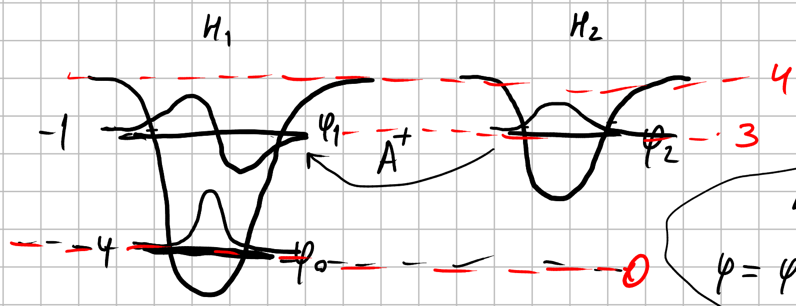
$$W(x) = \ell \cdot ln(ch x)$$

$$\varphi_0(x) = \frac{c}{(ch x)^\ell}$$

$$H_2 = -\partial_x^2 + \ell^2 th^2 x + \ell\frac{1}{ch^2 x} = -\partial_x^2 + \ell^2 - \frac{\ell(\ell-1)}{ch^2 x}$$

$\ell = 2$: $\quad H_1 = -\partial_x^2 + 4 - \frac{6}{ch^2 x}$

$$H_2 = -\partial_x^2 + 4 - \frac{2}{ch^2 x}$$



in susy:
$$\varphi = \varphi_1 + \psi\varphi_2$$

$$H_1 = A^+A \qquad A\varphi_0 = 0$$
$$H_2 = AA^+ \qquad (\partial_x + 2\, th x)\varphi_0 = 0$$

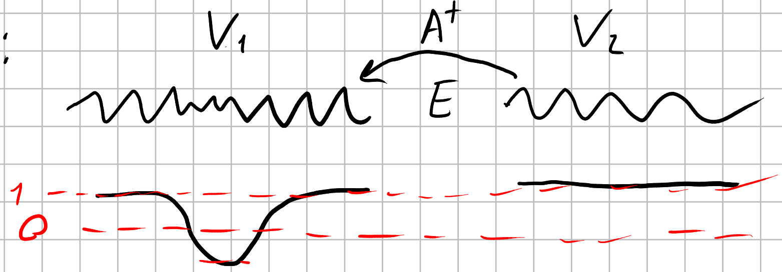
$$(\partial_x + th x)\varphi_2 = 0 \longrightarrow \varphi_2 = \frac{c_2}{ch x}$$

$$H_2\varphi_2 = 3\,\varphi_2$$

$$\varphi_0(x) = \frac{c}{(ch x)^2}$$

$$\varphi_1 = A^+\varphi_2 = (-\partial_x + 2\, th x)\frac{c_2}{ch x} = \frac{3c_2\, th x}{ch x}$$

$\underline{\ell = 1:}$



$$e^{ipx} \qquad p^2 = E - 1$$

$$A^+ e^{ipx} = (-\partial_x + \operatorname{th} x)\, e^{ipx} = (-ip + \operatorname{th} x)\, e^{ipx}$$

$$\Phi_E^{(2)} = e^{ipx} \begin{cases} \xrightarrow{-\infty} e^{ipx} \\ \xrightarrow{+\infty} e^{ipx} \end{cases} \qquad R_E^{(2)} = 0 \qquad T_E^{(2)} = 1$$

$$A^+ \Phi_E^{(2)} = (-ip + \operatorname{th} x)\, e^{ipx} \begin{cases} \xrightarrow{x \to -\infty} (-ip - 1)\, e^{ipx} \\ \xrightarrow{x \to +\infty} (-ip + 1)\, e^{ipx} \end{cases} \longrightarrow \begin{cases} e^{ipx} \\ \frac{ip-1}{ip+1}\, e^{ipx} \end{cases} \Phi_E^{(1)}$$

$$p^2 = E - 1$$

$$R_E^{(1)} = 0 \qquad T_E^{(1)} = \frac{ip - 1}{ip + 1} = e^{i\delta_T}$$