Tensor powers of vector representation of $U_q(sl_2)$ at even roots of unity

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$U_q(sl_2)$ at even roots of unity

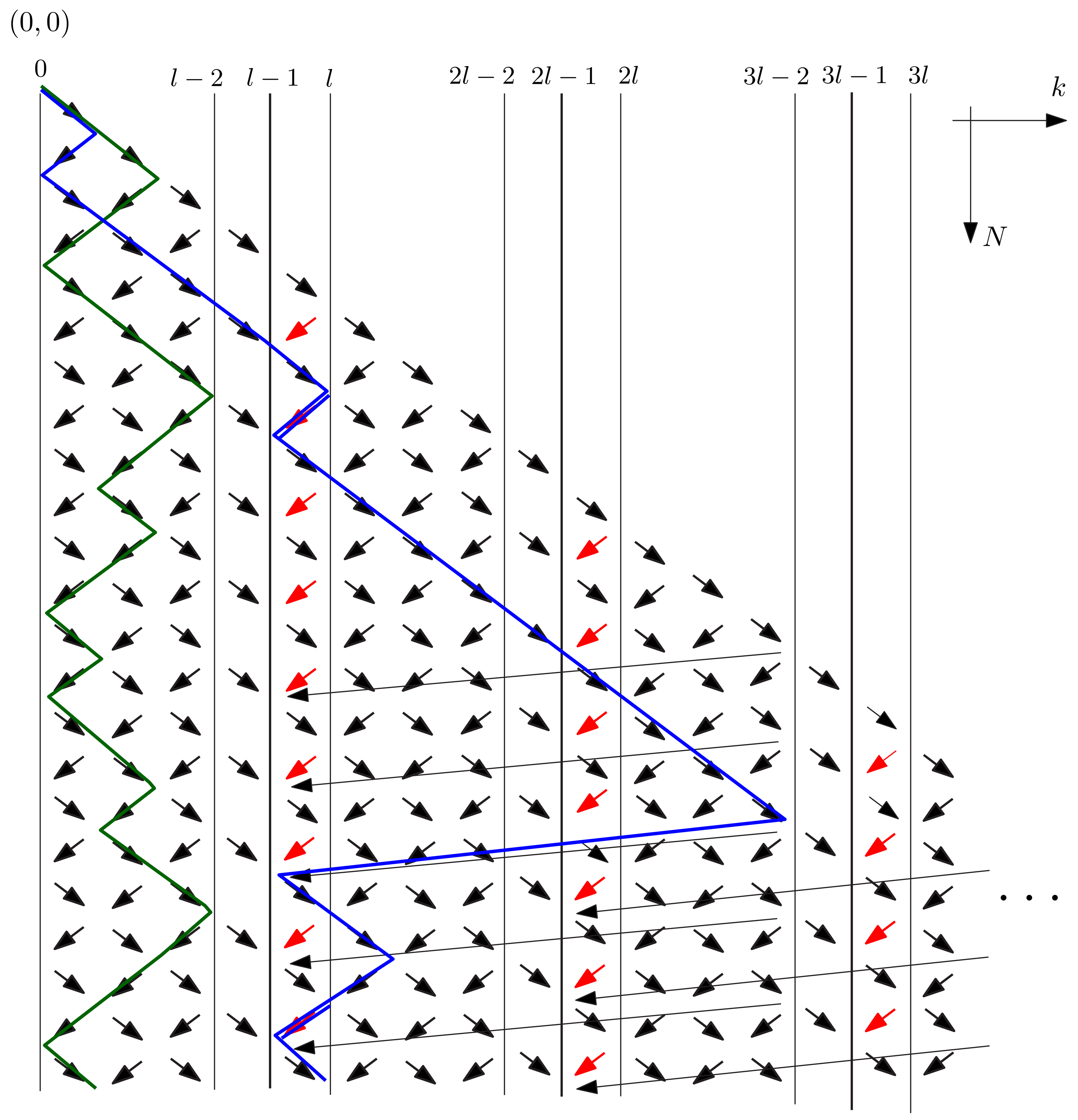
Consider Lustig quantum group of divided powers $U_q(sl_2)$ at even roots of unity $q = e^{\frac{\pi i}{l}}$, where l is odd. It is generated by E, $E^{(l)}$, F, $F^{(l)}$, $K^{\pm 1}$ and $\begin{bmatrix} K; 0 \\ l \end{bmatrix}$. with the relations

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}};$$

$$EK = q^2KE, \quad FK = q^{-2}KF, \quad KK^{-1} = K^{-1}K = 1;$$
$$K^{2l} = 1, \quad E^l = F^l = 0.$$

Recursion-induced lattice path model

Grothendieck ring of $\mathbf{Rep}(U_q(sl_2))$ induces recursion on multiplicity functions $M^{(l)}_{T(k)}(N)$. One can define lattice path model, where weighted numbers of paths descending from $(0,0)$ to (k, N) satisfy the same recursion.



Results and future directions

- We obtained formulas for multiplicities in tensor power decomposition of a fundamental representation for Lustig quantum group of divided powers $U_q(sl_2)$ and small quantum group $u_q(sl_2)$ at even roots of unity. We obtained the limit distributions for Plancherel measure and Character measure for the mentioned quantum groups in different asymptotic regimes.
- Statistics of tensor powers of other representations remain unexplored. Similar problem for $U_q(sl_n)$ is still open for both finite and infinite ranks. One would also need to give these findings interpretation with respect to the quantum Schur-Weyl duality, meaning, dimensions of what representations of Hecke algebra $nH_N(q)$ are being studied.

Tensor product decomposition

- Tensor power of fundamental representation decomposes as

$$T(1)^{\otimes N} = \bigoplus_{k=0}^{N} M^{(l)}_{T(k)}(N)T(k), \quad T(1), T(k) \in \mathbf{Rep}(U_q(sl_2))$$

- This decomposition allows one to define character measure

$$p_k^{(N,l)}(t) = \frac{M^{(l)}_{T(k)}(N)\mathrm{ch}T(k)(e^t)}{(\mathrm{ch}T(1)(e^t))^N}.$$

Counting paths

Counting numbers of paths in the recursion-induced lattice path model calls for definition of filter restrictions, regions, boundaries and congruence of regions. Latter allows one to obtain formula for multiplicities.

$$M^{(l)}_{T(k_1l+k_0)}(N) = F^{(N)}_{k_1l+k_0} + \sum_{j=1}^{\left[\frac{N-k_1l+1}{2l}+\frac{1}{2}\right]} F^{(N)}_{-k_1l+k_0-2jl} + \sum_{j=1}^{\left[\frac{N-k_1l+1}{2l}\right]} F^{(N)}_{k_1l+k_0+2jl},$$

$$F^{(N)}_M = \binom{N}{\frac{N-M}{2}} - \binom{N}{\frac{N-M}{2}-1}.$$

The limit shape

This formula for multiplicities allows one to obtain the limit shape for the character measure.
The highest weight is given by $k = k_1l+k_0$, the critical point of the large deviation rate function is denoted by ξ_0. In the regime, when $N = Ml$, $k_1 \to \infty$, $M \to \infty$, l, $\frac{k_1}{M} = \xi_0 + \frac{\alpha}{\sqrt{lM}}$.

$$p_k^{(N,l)}(t) = v_{k_0}\sqrt{\frac{1}{2\pi N}}(\mathrm{ch}T(1)(e^t))e^{-\frac{\alpha^2(\mathrm{ch}T(1)(e^t))^2}{8}},$$

where

$$v_{k_0} = \frac{(e^{t(k_0+1)} - e^{-t(k_0+1)})(e^{t(l-1-k_0)} - e^{-t(l-1-k_0)})}{e^{tl} - e^{-tl}}, \quad k_0 = 0, \ldots, l-2,$$

$$v_{l-1} = 1.$$

Vector $(v_0, \ldots, v_{l-1})$ is an eigenvector of the Markov matrix, corresponding to the random walk, induced by the tensor product decomposition, with eigenvalue $\lambda = 1$.

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