A NUMERICAL SOLUTION OF NONLOCAL PARABOLIC EQUATION .

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Abstract

We deals a diffusion equation with integral condition , this problem with non-local term which makes the numerical solution difficult. The existence of a weak solution in an appropriate sense as well as regularity results are obtained by using the Rothe method for the temporal discretization and the finite element method for the spatial variable.

problem Position

We consider an open bounded domain Ω of $\mathbb{R}^n$ with Lipschitz continuous boundary Γ. Fixing a final time $T \succ 0$, we set $I = [0, T]$ and $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}$,$\Gamma_D \cap \Gamma_N = \varnothing$ ($meas\,(\Gamma_D) \succ 0$) where Γ_D and Γ_N are tow open subsets of Γ and ν is the outword normal victor at each point of Γ. Our aim is to ivestigate the following inverse problem of identifying the missing Direchlet condition $\gamma\,(t)$

$$D^{\alpha} u\,(t,x) - a(l(u)) \triangle u\,(t,x) = f\,(t,x) \qquad \text{in } I \times \Omega \tag{P}$$

$$u\,(0,x) = u_0\,(x) \quad \text{in } \Omega \tag{P_a}$$

$$\nabla u.\nu = g, \text{ on } I \times \Gamma_N \tag{P_b}$$

$$u = \gamma\,(t) \ \text{ on } I \times \Gamma_D, \tag{P_c}$$

from an additionnal measurement of type

$$\int_{\Omega} I^{1-\alpha}\,(u\,(t,x))\,dx = \theta\,(t) \tag{P_d}$$

where $\alpha \in \,]0,1[$ .The fractional integral $I^{1-\alpha}$ and the derevative D^{α}_{RL} are understood here in Riemann-Liouville sense.

Physical Motivation

These problems apears in several areas of investigation, for example in science and engineering such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, biosciences, signal prrossecing, systems control theory, electrochemistry, mechanics and diffusion processes. For general motivations, relevant theory and its application . Fractional diffusion equations include the mathematical model of large class of problems. They describe anomalous diffusion on fractal (physical objects of fractional dimension), fractional random walk.

Variational formulation

The semi discrete formulation is

$$\begin{aligned} &\left(u_h^i, v_h\right) + \tau^{\alpha} a(l(u_h^i)) \left(\nabla u_h^i, \nabla v_h\right) + \tau^{\alpha} a(l(u_h^i)) \left(g^i, v_h\right)_{\Gamma_N} = \tau^{\alpha} f^i, v_h) + \left(u_h^{i-1}, v_h\right) \\ &+\tau^{\alpha} v_{h \setminus \Gamma_D} \left[\theta_i' + a(l(u_h^i)) \left(g^i, 1\right)_{\Gamma_N} - \left(f^i, 1\right)\right] \end{aligned} \tag{1}$$

A priori estimates

In this section, we derive some a priori estimates that will be used in showing existence of the solution.
Lemma 1 The following estimates

$$\sum_{i=1}^{l} \tau \parallel \delta u_i \parallel^2 \leq C, \ \ \|\nabla u_i\|^2 \leq C \qquad (2)$$

Lemma 2 There exists a positive constant C independent to i, j, n and τ such that

$$\int_I \|\partial_t I_n\|^2_{V^*} \leq C \qquad (3)$$

Existence results

definition 1 We say that (u, γ) is a weak solution of problem (P)if
$i)$ $u \in L^2\,(I, V)$ with $I^{1-\alpha}\,(u) \in C\,(I, V^*)$ and $\partial_t I^{1-\alpha}\,(u) \in L^2\,(I, V^*)$.
$ii)$ $u_{\setminus \Gamma_D} = \gamma\,(t)$
$iii)$ For any $\phi \in V$, we have

$$\begin{aligned} \int_I \left(\partial_t I^{1-\alpha}\,(u)\,, \phi\right) dt + \int_I a(l(u))\,(\nabla u, \nabla \phi)\,dt + \int_I a(l(u))\,(g, \phi)_{\Gamma_N}\,dt = \int_I (f, \phi)\,dt \\ + \int_I \phi_{\setminus \Gamma_D} \left[\theta' + a(l(u))\,(g, 1)_{\Gamma_N} - (f, 1)\right] dt \end{aligned}$$

Théorème

There exists $u \in L^2\,(I, V)$ and $\gamma \in L^2\,(0, T)$ such that $\{u, \gamma\}$ solves (P) in the sens definition(1).

Numerical Results

Now to illustrate the results of a simple numerical experiment.
We use Roth's approximation in time discretization and the finite element method for spatial discretization . We consider Ω as an interval, so we take for example $\Omega = (0,1)$, $T = (0,1)$, and the test solution $u(t,x) = xt(1 - \frac{1}{2}x)(1 - \frac{1}{2}t)$.
We take $\alpha = 1/2$, and $a(l(u)) = 1 + cos(d)$ with

$$\begin{aligned} f(t,x) &= \frac{1}{\sqrt{\Pi}}(x - x^2)(\frac{-4}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}}) - a(d)(-t + \frac{1}{2}t^2), \\ &\text{and} \\ \theta'(t) &= \frac{1}{\sqrt{\Pi}}(\frac{4}{9}t^{\frac{3}{2}} + \frac{2}{3}t^{\frac{1}{2}}). \end{aligned}$$

Numerical Results

For Newton's iteration, we consider initial guess u^0 as follows

$$u^0 = \begin{cases} 1, & \text{at interior node} \\ 0, & \text{at boundary node} \end{cases}$$

The numerical error is calculated first on u_{γ} in $L^2(0,T)$ and therefore we take N large enough in the following $N = 10^3$ and we assign the time step $\tau = \frac{1}{10k}$, $k = 1, 2...6$ .The $table1$ gives the numerical errors and $Fig1$. shows the results of error in loglog-plot.

Numerical Results

Table 1.

τ	$\|u(t,1) - u^{(N)}\|_{L^2([0,T])}$
$\frac{1}{10}$	$4.2866e-003$
$\frac{1}{20}$	$2.2108e-003$
$\frac{1}{30}$	$1.4928e-003$
$\frac{1}{40}$	$1.0800e-003$
$\frac{1}{50}$	$8.9679e-004$
$\frac{1}{60}$	$7.4682e-004$

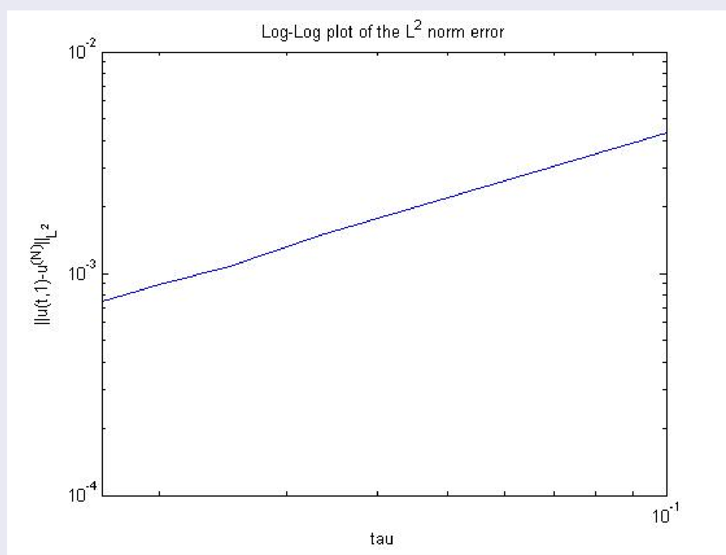

FIGURE:

the bibliography

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