Applications of the motivic Becker-Gottlieb transfer

following Carlsson-Joshua

joint work with R. Joshua

Interlude, classical construction

Becker–Gottlieb

G-equivariant bundles

A. Dold

In terms of fixed points

e.g. $V \xrightarrow{\pi} X$ vector bundle.

$$\begin{tikzcd} Z_s \arrow[r] \arrow[d] & X \arrow[d, "s_0"] \\ X \arrow[r, "s"] & V \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

"fixed points" = Locus of zeros of s.

zero section

generic section

transfer

$$\Sigma^\infty X_+ \xrightarrow{tr} \Sigma^\infty Z_s$$

Lewis, May, Steinberger

G-equivariant spectra

Group G reductive (in the sense of SGA III)

linearly reductive is not needed.

Special GL_n

Non-special $O(n)$, fin. groups

$\hookrightarrow$ k infinite

3 basic contexts

1) $\begin{array}{c} X \\ \downarrow \\ R \end{array}$ smooth $\quad f : X \to X$ G-equivariant

2) $E \to B$ $\;G$-torsor

a) $\pi_Y : E \times_G (Y \times X) \longrightarrow E \times_G Y$

b) $BG^{gm,m}$ aprox to BG

$$\begin{array}{c} BG^{gm,m} \\ \pi_m \uparrow \\ E^{gm,m} \end{array}$$

Totaro, Morel-Voevodsky

$\pi_Y : E^{gm,m} \times_G (Y \times X) \to E^{gm,m} \times_G Y$

(c) $\operatorname{colim} P_m$:

$$EG \times_G (Y \times X) \longrightarrow EG \times_G Y$$

Think choices $EG^{gm,m}$

$$\begin{tikzcd} EG^{gm,m} & , & E{G'}^{gm,m} \\ & EG^{gm,m} \times E{G'}^{gm,m} \arrow[ul] \arrow[ur] & \end{tikzcd}$$ ✓

Apply the slice filtration.

$m \to \infty$

slice connectivity increases.

$\exists$ complete w.r.t. slice tower

Thm (Carlsson–Joshua)

$$\exists\ tr(f_y): \Sigma_T E\times_G Y_+ \to \Sigma_T E\times_G (Y\times X)_+$$

$$\begin{tikzcd}
\Sigma_T E\times_G Y_+ \arrow[r, "tr(f_y)"'] & \Sigma_T E\times_G (Y\times X)_+ \arrow[l, bend right, "\pi_y"'] \arrow[d] \\
\Sigma_T \mathrm{Spec}\,k \arrow[u] \arrow[r, "\tau_y"'] & \Sigma_T \mathrm{Spec}\,k
\end{tikzcd}$$

this map comes from duality

trace

s.t.

- projection formula
- naturality

$$\alpha \in h^{*,*}(E\times_G Y),\ \beta \in h^{*,*}(E\times_G Y\times X)_+$$

$$tr(f_y)^*(\pi_y^*(\alpha)\cdot\beta)$$
$$\|$$
$$\alpha\cdot tr(f_y)^*\beta$$

Thm (J. P.) Additivity for trace

$\longrightarrow$ transfer

(i) $X = X_1 \cup X_2$ Zariski open.

$U \hookrightarrow X$ open

$U_i = X_i \cap U$

$$\tau_{X/U} = \tau_{X_1/U_1} + \tau_{X_2/U_2} - \tau_{X_1 \cap X_2 / U_1 \cap U_2}$$

$\otimes \mathbb{Z}\left[\frac{1}{p}\right]$ if char $p > 0$

(ii) $i: Z \longrightarrow X$ closed G-embedding

$$\Rightarrow \tau_X = \tau_U + \tau_{X/U}, \quad U = X \setminus Z$$

$Z, X \in Sm_k$

transfer

$\tau_{X/U} = \tau_Z$ up to a unit $GW(k)$

Moreover, $\sqrt{-1} \in k$

$\Longrightarrow$ trace $-\wedge E$

E rigid.

$$I_{X/U} = I_{Th(N_L)} = I_Z$$

Borel construction

$\otimes \mathbb{Z}[\frac{1}{p}]$ if char $p > 0$

(3) $\{S_\alpha, \alpha\}$ stratification of X

$\hookrightarrow$ locally closed & smooth

$$\Longrightarrow I_X = \sum_\alpha I_{S_\alpha}$$

D. Cox

Etale tubular nbds.

assuming $\sqrt{-1} \in k$.

transfers after $-\wedge E$

Conjecture (Morel)

↳ Some cases (Levine)

Thm G split linear algebraic group over a perfect field k s.t. $\sqrt{-1} \in k$. $N(T)$ normalizer of a split max. torus in G.

$$\Longrightarrow$$

$$\tau_{G/N(T)}(1) = 1 \in GW(k)$$

$- \otimes \mathbb{Z}[\frac{1}{p}]$ char $p > 0$

• $\tau_{G/T}(1) = |W|$ $W = N(T)/T$

$GW(k)$

Geometric input:

(Thomason)

T split torus acting on $X \in Sm_k$

k = perfect

$\Longrightarrow$

$X = \coprod X_j$ locally closed $\subseteq X$

T-stable

s.t. $X_j \underset{\text{T-schemes}}{\xrightarrow{\cong}} (T/\Gamma_j) \times Y_j$

$\Gamma_j \subseteq T$ subgroup, Y_j regular.

T acts trivially on Y_j

Thm. (J. P.)

1) $\tau_{\mathbb{G}_m} = 0$, T split torus

$$\tau_T = 0$$

2) T split torus acting on $X \in Sm_k$

$\Rightarrow X^T \in Sm_k$

and $\tau_X = \tau_{X^T}$

Pf. 1) $\tau_{\mathbb{G}_m} = 0$ from

M.U. trace and $\mathbb{A}^1$-invariance

$\tau_T = 0$ multiplicative properties

2) Thomason + additivity on strata

$$X_j = (T/\Gamma_j) \times Y_j$$

trivial action

$X^T = \coprod X_{j'}$, $\Gamma_{j'} = T$

smooth

addity of τ: $\Gamma_j \neq T$

$$\tau_{T/\Gamma_j \times Y_j} = 0$$

$\tau_{T/\Gamma_j} = 0$ previous case

$\boxtimes$

Pf (Morel's conjecture)

G connected $\qquad G = SL_2$
$\qquad\qquad\qquad\qquad \cup\!\!\!\!\, | \quad T$

$G/N(T) \curvearrowleft T$

↳ variety of all split max. tori in G

$(G/N(T))^T = $ class of $1 \cdot N(T)$
$\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad \{\operatorname{Spec} k\}$

↳ Brion, Peyre

Thm $\Rightarrow$

$$\tau_{G/N(T)}(1) = \tau_{(G/N(T))^T}(1)$$
$$= \tau_{\operatorname{Spec} k}(1) = 1$$

$\boxtimes$

$$G/N(T) \underset{\text{open}}{\subseteq} \mathrm{Sym}\,\mathbb{P}^1 = \mathbb{P}^2$$

Double coset formula

G linear algebraic group

$H, K \subseteq G$ closed linear subgroups

$$\begin{tikzcd}
EK \times_K G/H \arrow[r, "\overline{p_K}"'] \arrow[d, "\overline{\pi_H}"'] & EG \times_G G/H \arrow[d, "\pi_H"] \\
BK \arrow[r, "p_K"'] & BG
\end{tikzcd}$$

$K \curvearrowright G/H = \coprod_i F_i \rightarrow$ loc. closed $\subseteq G$, K-stable smooth

$\uparrow$
H

Thm 1) $p_K^* \circ tr_G^* = tr_K^* \circ p_K^*$

2) $\sqrt{-1} \in k$ E rigid ✓

$\Longrightarrow$

$$h^{*,*}(tr_K, E) = \sum_j h^{*,*}(\iota_j \circ tr^K_{F_j}, E)$$

$\iota_j : EK \times_K F_j \longrightarrow EK \times_K G/H$

induced by $F_j \to G/H$

additivity for tr_G $\boxtimes$