Euler International Mathematical Institute

St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences
and St. Petersburg State University

# SPECTRAL THEORY AND MATHEMATICAL PHYSICS

**Asymptotic Methods of Mathematical Physics (Buslaev conference).**
**12th St. Petersburg conference in Spectral Theory (Birman conference).**
**Summer School.**

20 – 30 June 2021

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## *Program and abstracts*

St. Petersburg, 2021

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Alexander Fedotov, Nikolai Filonov, Alexander Its, Ilya Kachkovskiy,
Alexander Sobolev, Tatyana Suslina, Dmitri Yafaev

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Spectral Theory and Mathematical Physics.
**Asymptotic Methods of Mathematical Physics (Buslaev conference). 12th St. Petersburg conference in Spectral Theory (Birman conference). Summer School.**
Program and abstracts. Euler International Mathematical Institute, St. Petersburg, 2021.

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Program website: https://indico.eimi.ru/e/STMP21

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# PROGRAMS

## Buslaev conference (June, 20 – June, 22)

**SUNDAY 20 June:**

12:30–13:00: REGISTRATION

13:00–15:00: Lunch

15:00–15:50: Alexander Its (Indiana University – Purdue University Indianapolis, USA, and St. Petersburg State University, Russia). *Isomonodromy aspects of the tt* equations of Cecotti and Vafa. Iwasawa factorization and asymptotics.*

15:50–16:05: Coffee break

16:05–16:55: Maxim Skriganov (St. Petersburg Department of Steklov Mathematical Institute, Russia). *Point distributions in compact Riemannian manifolds.*

17:00–17:50: Alexander Fedotov (St. Petersburg State University, Russia). *Monodromization for Schrödinger operators with meromorphic potentials.*

17:50–18:10: Coffee break

18:10–19:00 Leonid Pastur (B. Verkin Institute for Low Temperature Physics and Engineering, Ukraine). *Entanglement entropy of free disordered fermions and spectral theory* **(online talk).**

19:10–20:00: Estelle Basor (American Institute of Mathematics, USA). *Factoring Fredholm Determinants* **(online talk).**

**MONDAY 21 June:**

11:00–12:00: Free coffee and free discussions

12:00–12:50: Vladimir Nazaikinski (Ishlinsky Institute for Problems in Mechanics RAS, Russia). *Efficient semiclassical asymptotics.*

13:00–15:00: Lunch

15:00–15:50: Sergei Dobrokhotov (Ishlinsky Institute for Problems in Mechanics RAS, Russia). *Homogenization, adiabatic approximation and pseudodifferential operators.*

15:50–16:10: Coffee break

16:10–17:00: Mikhail Lyalinov (St. Petersburg State University, Russia). *Asymptotics for the eigenfunctions of Laplacians in some unbounded domains with Robin-type boundary conditions and functional equations.*

17:10–18:00: Percy Deift (Courant Institute of Mathematical Sciences, USA). *The open Toda chain with external forcing* **(online talk).**

18:30–21:00: Boat trip with a buffet

**TUESDAY 22 June:**

10:00–10:50: Viktor Novokshenov (Institute for Mathematics UFRC RAS). *Autoresonance in a model of a terahertz wave generator.*

10:50–11:10: Coffee break

11:10–12:00: Vladimir Sukhanov (St. Petersburg State University, Russia). *Riemann–Hilbert approach to the inverse problem for the Schrödinger operator with quadratic potential.*

12:10–13:00: Andrei Prokhorov (University of Michigan and St. Petersburg State University). *Integrable structure for the multipoint distribution of TASEP.*

13:00–15:00: Lunch

15:00–15:50: Mikhail Belishev and Dimitrii Korikov (St. Petersburg Department of Steklov Mathematical Institute, Russia). *Recent results on Electric Impedance Tomography of* 2*-dim Riemannian manifolds.*

15:50–16:10: Coffee break

16:10–17:00: Alexander Minakov (Charles University, Czech Republic). *Asymptotic analysis for step-like problems for integrable equations* **(online talk)**.

# Birman conference (June, 23 – June, 26)

**WEDNESDAY 23 June:**

9:30–10:00: Registration

10:00–10:50: Andrey Piatnitski (The Arctic University of Norway, UiT, campus Narvik and IITP RA). *Homogenization of Steklov sieve.*

10:50–11:10: Coffee break

11:10–12:00: Tatiana Suslina (St. Petersburg State University). *Homogenization of the periodic Schrödinger-type equations.*

12:10–13:00: Sergei Nazarov (St. Petersburg State University, Russia). *Threshold resonances and virtual levels in the spectrum of periodic waveguides.*

13:00–15:00: Lunch

15:00–15:50: Valery Smyshlyaev (University College London, UK). *Uniform asymptotics for families of degenerating variational problems and applications to error estimates in homogenization.*

15:50–16:10: Coffee break

16:10–17:00: Nikita Senik (St. Petersburg State University, Russia). *On homogenization for locally periodic elliptic problems on a domain.*

**THURSDAY 24 June:**

10:00–10:50: Andrei Shafarevich (Moscow State University). *Semiclassical eigenvalues for the Schrödinger operators on surfaces with conic points and with $\delta$-potentials.*

10:50–11:10: Coffee break

11:10–12:00: Andrey Shkalikov (Moscow State University). *Spectral properties of ordinary differential operators generated by a first order system.*

12:10–13:00: Denis Borisov (Institute of Mathematics, Ufa, Federal Research Center, RAS). *On bifurcations of internal thresholds in essential spectrum under small non-symmetric perturbations.*

13:00–15:00: Lunch

15:00–15:50: Alexey Karapetyants (Southern Federal University). *Mixed norm spaces of analytic functions as spaces of generalized fractional derivatives of functions in Hardy type spaces.*

15:50–16:10: Coffee break

16:10–17:00: Igor Sheipak (Lomonosov Moscow State University). *Spectral properties of a singular string equation: continuous spectrum and eigenvalues.*

17:30–20:00: Boat trip with a buffet

**FRIDAY 25 June:**

10:00–10:50: Nikolai Filonov (PDMI RAS and St. Petersburg State University, Russia). *On the rate of decrease at infinity of solutions to a Schrödinger equation in a half-cylinder.*

10:50–11:10: Coffee break

11:10–12:00: Ilya Kachkovskiy (Michigan State University). *Ballistic transport for one-dimensional quasiperiodic Schrödinger operators.*

12:10–13:00: Elena Zhizhina (IITP RAS). *Ground state for nonlocal Schrödinger operator and spatially inhomogeneous contact models.*

13:00–15:00: Lunch

15:00–15:50: Alexander Nazarov (PDMI RAS and St. Petersburg State University). *The structure of the Dirichlet–Laplacian spectrum in the Fichera layers and crosses of arbitrary dimension.*

15:50–16:10: Coffee break

16:10–17:00: Andrew Comech (Texas A&M University, College Station, Texas and IITP, Moscow, Russia). *Virtual levels of operators in Banach spaces and application to Schrödinger operators.*

17:10–18:00: Vladimir Kapustin (St. Petersburg Department of the Steklov Mathematical Institute). *The set of zeros of the Riemann zeta function as the point spectrum of an operator.*

**SATURDAY 26 June:**

9:40–10:05: Vladimir Bobkov (Institute of Mathematics UFRC RAS). *On multiplicity properties of higher eigenvalues of the p-Laplacian.*

10:10–10:35: Dmitry Polyakov (Southern Mathematical Institute of Vladikavkaz Scientific Center of RAS). *Spectral asymptotics for a fourth-order differential operator with multipoint boundary conditions.*

10:40–11:00: Coffee break

11:00–11:25: Anna Tsvetkova (Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences). *Asymptotics in the form of special functions of eigenfunctions of the operator* $\nabla D(\mathbf{x})\nabla$ *defined in a two-dimensional domain and degenerating on its boundary.*

11:30–11:55: Anna Allilueva (Ishlinsky Institute for Problems in Mechanics RAS). *Asymptotic solutions for nonlinear equations of gas dynamics, describing smoothed discontinuities.*

12:00–13:30: Lunch

13:30: Trip to Kronstadt and Peterhof

# Summer school (June, 27 – June, 30)

**SUNDAY 27 June:**

9:30–10:00: Registration

10:00–10:45 and 10:50–11:35: Dimitri Yafaev (Université de Rennes and St. Petersburg State University). *Spectral theory of Jacobi operators and asymptotic behavior of orthogonal polynomials.*

11:35–11:55: Coffee break

11:55–12:40 and 12:45–13:30: Vladimir Nazaikinskii (Ishlinsky Institute for Problems in Mechanics RAS, Russia). *Spectral flow and some applications.*

13:30–15:15: Lunch

15:15–16:00 and 16:05–16:50: Denis Borisov (Institute of Mathematics, Ufa Federal Research Center, RAS, Russia). *Eigenvalues and resonances emerging from thresholds in essential spectra.*

16:50–17:10: Coffee break

17:10–17:35: Alexey Kosarev (Moscow State University, Russia). *Asymptotics of fundamental solutions to* $2 \times 2$ *first order system of ordinary differential equations.*

17:40–18:05: Tatiana Garmanova (Moscow State University, Russia). *Embedding constants in Sobolev spaces.*

**MONDAY 28 June:**

10:00–10:45 and 10:50–11:35: Dimitri Yafaev (Université de Rennes and St. Petersburg State University). *Spectral theory of Jacobi operators and asymptotic behavior of orthogonal polynomials.*

11:35–11:55: Coffee break

11:55–12:40 and 12:45–13:30: Vladimir Nazaikinskii (Ishlinsky Institute for Problems in Mechanics RAS, Russia). *Spectral flow and some applications.*

13:30–15:15: Lunch

15:15–16:00 and 16:05–16:50: Denis Borisov (Institute of Mathematics, Ufa Federal Research Center, RAS, Russia). *Eigenvalues and resonances emerging from thresholds in essential spectra.*

16:50–17:10: Coffee break

17:10–17:35: Konstantin Zhuikov (RUDN University, Moscow, Russia). *Eta-invariant for parameter-dependent families with periodic coefficients.*

18:00–20:30: Boat trip with a buffet

**TUESDAY 29 June:**

10:00–10:45 and 10:50–11:35: Andrey Piatnitski (The Arctic University of Norway, UiT, campus Narvik and IITP RAS). *Stochastic homogenization of convolution type operators and convolution type energies.*

11:35–11:55: Coffee break

11:55–12:40 and 12:45–13:30: Alexander Fedotov (St. Petersburg State University). *A short introduction to the theory of ergodic operators.*

13:30–15:15: Lunch

15:15–16:00 and 16:05–16:50: Valery Smyshlyaev (University College London, UK). *High-frequency scattering by boundary inflection: a model for asymptotic transition from discrete to continuous.*

16:50–17:10: Coffee break

17:10–17:35: Olga Shchegortsova (Moscow Institute of Physics and Technology (National Research University)). *Gaussian beam solutions to the Cauchy problem for the Schrödinger equation with a delta potential.*

17:40–18:05: Vasily Sergeev (St. Petersburg State University). *On adiabatic evolution generated by a one-dimensional Schrödinger operator.*

**WEDNESDAY 30 June:**

10:00–10:45 and 10:50–11:35: Andrey Piatnitski (The Arctic University of Norway, UiT, campus Narvik and IITP RAS). *Stochastic homogenization of convolution type operators and convolution type energies.*

11:35–11:55: Coffee break

11:55–12:40 and 12:45–13:30: Alexander Fedotov (St. Petersburg State University). *A short introduction to the theory of ergodic operators.*

13:30–15:15: Lunch

15:15–16:00 and 16:05–16:50: Valery Smyshlyaev (University College London, UK). *High-frequency scattering by boundary inflection: a model for asymptotic transition from discrete to continuous.*

16:50–17:10: Coffee break

# Abstracts of the talks

## Buslaev conference

### Isomonodromy aspects of the tt* equations of Cecotti and Vafa. Iwasawa factorization and asymptotics

Alexander Its
Indiana University – Purdue University Indianapolis and
St. Petersburg State University

In this talk the results concerning the global asymptotic analysis of the tt* - Toda equation,

$$2(w_i)_{t\bar{t}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i+1})},$$

where, for all $i$, $w_i = w_{i+4}$ (periodicity),$w_i = w_i(|t|)$ (radial condition), and $w_i + w_{-i-1} = 0$ ("anti-symmetry"), will be presented. The problem has been studied in the early 90s by Cecotti and Vafa in connection with their classification of supersymmetric field theories, and a series of important conjectures about the solutions has been formulated.

We study the question using a combination of methods from p.d.e., isomonodromic deformations (Riemann–Hilbert method), and loop groups (Iwasawa factorization). We place these global solutions into the broader context of solutions which are smooth near 0. For such solutions, we compute explicitly the Stokes data and connection matrix of the associated meromorphic system, in the resonant cases as well as the nonresonant case. This allows us to give a complete picture of the monodromy data, holomorphic data, and asymptotic data of the global solutions.

This is a joint work with Martin Guest and Chang-Shou Li.

### Point distributions in compact Riemannian manifolds

Maxim Skriganov
St. Petersburg Department of Steklov Mathematical Institute

We consider finite point subsets (distributions) in compact connected metric measure spaces. The spaces under study are specialized by conditions on the volume of metric balls as a function of radii. These conditions are not hard and hold, particularly, for all compact Riemannian manifolds. Under these conditions we prove nontrivial upper bounds for the $L_p$-discrepancies of point distributions for any $p > 0$ and $p = \infty$. The order of these bounds is sharp, at least, for compact Riemannian symmetric manifolds of rank one and $2 \leq p < \infty$.

The talk is based on the papers:

[1] M.M.Skriganov, Bounds for $L_p$-Discrepancies of Point Distributions in Compact Metric Measure Spaces, Constructive Approximation (2019), Volume 51, pp 413-425;
[2] M.M.Skriganov, Point Distributions in Two-Point Homogeneous Spaces, Mathematika (2019), Volume 65(3), pp 557-587.

# Monodromization for Schrödinger operators with meromorphic potentials

Alexander Fedotov
St. Petersburg State University

Monodromization method is a renormalization approach that was developed by V. Buslaev and A. Fedotov when trying to use ideas of the Bloch-Floquet theory to study solutions to difference equations of the form

$$\Psi(x+h) = M(x)\,\Psi(x), \quad x \in \mathbb{R}, \tag{1}$$

where $h \in (0,1)$ is a fixed number, and $M$ is a 1-periodic $SL(2,\mathbb{C})$-valued function.

The usual Bloch–Floquet theory can be used, say, to study solutions to the differential equation

$$\frac{d\Psi}{dx}(x) = M(x)\,\Psi(x), \quad x \in \mathbb{R}. \tag{2}$$

A monodromy matrix is the matrix of the shift operator $s_1 : f(\,\cdot\,) \mapsto f(\,\cdot\,+1)$ restricted to the solution space. The space of vector-valued solutions to (2) being a two-dimensional linear space, a monodromy matrix is a $2 \times 2$ matrix independent of $x$. Given a fundamental matrix solution $\Psi$ to (1), one defines the corresponding monodromy matrix $M_1$ by the relation

$$\Psi(z+1) = \Psi(z)\,M_1^t,$$

where $\cdot^t$ denotes transposition. If the matrix $M_1$ is diagonalizable, one can easily construct Bloch solutions to (2), i.e., solutions that are eigenfunctions of the shift operator $s_1$

The space of vector-valued solutions to (1) is a two-dimensional modul over the ring of $h$ periodic functions. As a result a monodromy matrix is a $2 \times 2$-valued $h$-periodic matrix function. When defining it, it is convenient to make a linear change of variables for this matrix become 1-periodic.

It appears that, trying to construct Bloch solutions to (1), one comes to one more difference equation

$$\Psi_1(x+h_1) = M_1(x)\,\Psi_1(x), \quad x \in \mathbb{R}, \tag{3}$$

where $h_1 = \{1/h\}$ is a Gauss transform of $h$, and $M_1$ is a monodromy matrix for the input equation (1). The passage from (1) to (3) is called monodromization.

Equation (3) being similar to the input one, one can make one more monodromization and so on. This leads to an infinite sequence of equations of the form (1). The dynamical system defined by the map $(h,\, M) \mapsto (h_1, M_1)$ determines properties of solutions to the input equation at infinity.

The monodromization idea was successfully used to study the cantorien structure of the spectrum of the almost Mathieu operator in the semiclassical case, and to study spectral properties of one-dimensional differential quasiperiodic operators with two periods, one being large with respect to the other.

One of the important results of Buslaev and Fedotov was that certain sets of matrix-valued trigonometric polynomials are invariant with respect to the monodromization. In my talk, I concentrate on the case where $M$ is meromorphic, has a finite number of poles per period and is bounded at infinity.

The research was supported by Russian Science Foundation grant no. 17-11-01069.

## Entanglement entropy of free disordered fermions and spectral theory

Leonid Pastur

B. Verkin Institute for Low Temperatures Physics and Engineering, Ukraine

We present an overview of recent results on asymptotic trace formulas for finite-difference operators. The results are motivated by the problem on the large-block behavior of the entanglement entropy of free fermions arising in quantum statistical mechanics and quantum informatics. We begin with formulating a quite general setting that includes the problem as well as Szegö's problem on the asymptotic behavior of the determinant of the Töplitz (or discrete convolution) operators. In the frameworks of the setting we discuss a collection of recent results on the asymptotic form of the entanglement entropy of disordered free fermions whose one-body Hamiltonian is the discrete Schrödinger operator with ergodic (e.g. random or almost periodic). We present a variety of asymptotic formulas including those different from Szegö's theorem and determined by the smoothness of the test function and the symbol as well as the amount of randomness of the potential and the spectral type of the corresponding operator.

## Factoring Fredholm Determinants

Estelle Basor

American Institute of Mathematics

This talk will discuss exact identities for determinants of classes of structured matrices, including finite Toeplitz and finite Toeplitz plus Hankel. The identities are generally employed to find the asymptotics for such determinants. The talk will describe how the identities in certain cases for Toeplitz matrices can be factored into terms that arise in other identities and applications of the factoring to number theory.

## Efficient semiclassical asymptotics

Vladimir Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS

The canonical operator created by V. P. Maslov [Mas] in 1965 is one of the most powerful tools for constructing global semiclassical asymptotics. It is based on a geometric object—a Lagrangian manifold $\Lambda$ in the phase space $\mathbb{R}^{2l}_{(x,p)}$. The canonical operator acts on smooth functions on $\Lambda$ and transforms them into rapidly oscillating functions depending on the small parameter $h > 0$ on the physical space $\mathbb{R}^{l}_{x}$. In practical calculations, it is important that the asymptotics be efficient, i.e., allow the problem to be studied rather quickly and with rather modest computational costs. The concept of efficiency depends on the computational tools available and has changed significantly with the advent of technical computing systems *Wolfram Mathematica*, *MatLab*, and the like, which provide fundamentally new opportunities for the operational implementation and visualization of mathematical constructions. The classical definition of the canonical operator does not always meet these requirements, partly because the local coordinates it uses on the Lagrangian manifold are a priori not related in any way to the problem to be solved. The talk gives an overview of modern versions and modifications of the canonical operator allowing one to construct efficient asymptotics. The corresponding constructions include

• New formulas that permit writing the canonical operator in arbitrary coordinates on a Lagrangian manifold.

• Representations of the canonical operator in a neighborhood of generic caustics in the form of special functions (Airy, Pearcey, etc.) of a composite argument.

• In problems with localized initial data, generalizations of the canonical operator that allow one to study the cases in which the effective Hamiltonians (and, as a consequence, the Lagrangian manifolds) have a singularity of a special form at $p = 0$ (this includes, for example, the wave equation, Petrovskii hyperbolic systems, a pseudodifferential equation describing water waves in the linearized approximation taking into account dispersion, etc.).

The report is based on the results of many years of joint work by S.Yu.Dobrokhotov, A.I.Shafarevich and the author, which are mainly summarized in [DNS], [DN], and the survey [UMN].

[Mas] V. P. Maslov, *Perturbation Theory and Asymptotic Methods*, Izd. MGU, Moscow, 1965. (Russian)

[DNS] S. Yu. Dobrokhotov, V. E. Nazaikinskii, and A. I. Shafarevich, *New integral representations of the Maslov canonical operator in singular charts*, **Izv. RAN. Ser. Mat.**, 81, no. 2 (2017), 53–96; **Izv. Math.**, 81, no. 2 (2017), 286–328.

[DN] S. Yu. Dobrokhotov and V. E. Nazaikinskii, *Lagrangian manifolds and efficient short-wave asymptotics in a neighborhood of a caustic cusp*, **Mat. Zametki**, 108, no. 3 (2020), 334–359; **Math. Notes**, 108, no. 3 (2020), 318–338.

[UMN] S. Yu. Dobrokhotov, V. E. Nazaikinskii, and A. I. Shafarevich, *Efficient asymptotics of solutions of the Cauchy problem with localized initial data for linear systems of differential and pseudodifferential equations*, **Uspekhi Matem. Nauk** (2021) (in press).

## Homogenization, adiabatic approximation and pseudo differential operators

Sergey Dobrokhotov

Ishlinsky Institute for Problems in Mechanics RAS

More than 30 years ago, almost simultaneously, V. S. Buslaev and S. Yu. Dobrokhotov pointed out the connection between asymptotic methods for solving linear equations with rapidly changing coefficients and the adiabatic approximation. From the point of view of semiclassical approaches, the adiabatic approximation, in turn, is embedded in the theory of equations with an operator-valued symbol, in solving which it is very convenient to use Feynman-Maslov operator calculations. We discuss this approach, its development, its connection with homogenization, and its applications to some problems of quantum mechanics and continuum mechanics.

## Asymptotics for the eigenfunctions of Laplacians in some unbounded domains with Robin-type boundary conditions and functional equations

Mikhail Lyalinov

St. Petersburg University

In this report we study eigenfunctions of discrete or essential (continuous) spectrum for a Laplacian in domains that have wedge- or cone-shaped boundaries. The boundary conditions of the Robin type are valid and simulate various physical situations (quantum graphs, diffraction by semi-transparent or impedance boundaries, linear oscillations of fluid in wedges etc.) Asymptotics of the eigenfunctions at far distances are addressed.

We consider two examples. The first one deals with 2D diffraction of a plane (or surface) wave in a polygonal domain. The corresponding solution of the problem at hand represents an eigenfunction of (absolutely) continuous spectrum. Its asymptotics is efficiently computed and also some numerical results for the scattering amplitude are given. The method of the study is based on reduction of the problem to a system of functional equations by means of special (Sommerfeld) integral representation. The system of functional equations is then reduced to an integral equation with the integral operator analytically depending on the wave number. Then analytic Fredholm alternative is exploited to study the integral equation. The asymptotics is derived in a traditional manner by use of the asympotic evaluation of the integral representation for the eigenfunction. The asymptotics obtained is uniform with respect to the angle of observation.

In the second problem we consider the asymptotic behaviour for the eigenfunctions of the discrete spectrum of a Schrödinger operator in 3D space with singular $\delta'$-potential having its support on a circular conical surface. By use of the Kontorovich-Lebedev (KL) transform we reduce the spectral problem to that for the functional equation depending on the spectral parameter. Further reduction to an integral equation enables one to study the eigenfunctions of the corresponding integral operator. The latters are transformed accordingly and are then substituted to the KL integral representation. The asymptotics at far distances of the KL integral is computed by reduction to the Sommerfeld integral. The exponentially decaying asymptotics of an eigenfunction is obtained. However, the rate of the exponential decay depends on direction of observation. It is shown that in a close vicinity of some conical surface the asymptotics is described by a special function, namely, cylindrical (or Weber) function. Small angular neighbourhood of this surface plays the role of the transition zone, where the exponential decay is switched from one rate to another.

## The open Toda chain with external forcing

Percy Deift

Courant Institute of Mathematical Sciences

We consider the open Toda chain with external forcing, and in the case when the forcing stretches the system, we derive the longtime behavior of solutions of the chain. Using an observation of Jürgen Moser, we then show that the system is completely integrable, in the sense that the 2N-dimensional system has N functionally independent Poisson commuting integrals, and also has a Lax-Pair formulation. In addition, we construct action-angle variables for the flow. In the case when the forcing compresses the system, the analysis of the flow remains open.

## Autoresonance in a model of a terahertz wave generator

Victor Novokshenov

Institute of Mathematics UFRC RAS

We study a model of an electromagnetic wave generator based on a system of coupled Josephson junctions. The model is a chain of coupled sine-Gordon equations for the phases of the electric field in the junctions under dissipation and constant pumping. We find conditions for a resonant field excitation under various parameters of the system. It turns out that the chain of sine-Gordon equations evokes an autoresonance with a certain dependence of the frequency on the magnitude of the Josephson pumping current. We construct an asymptotic expansion for a solution of the chain under a large resonant frequency. The leading terms of the expansion for the phases of the electric field are linear in time, which is typical of an autoresonance in a system of coupled oscillators. The key role here is played by the main resonance equation, which defines the mode of the resonant excitation of the chain. This is the equation of a mathematical pendulum with periodically changing mass. A class of solutions of this equation is studied in detail as well as classes of separatrix solutions corresponding to zero initial velocity of the pendulum. It is proved that there exists a separatrix $\pi$-kink type solution on which the autoresonance mode is generated in the original chain of sine-Gordon equations.

The talk is based on the joint paper with Oleg M. Kiselev.

## Riemann–Hilbert approach to the inverse problem for the Schrödinger operator with quadratic potential

Vladimir Sukhanov

St. Petersburg University

The talk is concerned with the direct and the inverse spectral problems for the operator $L = -\frac{d^2}{dx^2} - \frac{x^2}{4} + v(x)$ on the line with the potential $v(x)$ from the Schwartz class. We consider analogs of Jost solutions for this operator and construct corresponding Riemann–Hilbert problem. We prove solvability of the Riemann–Hilbert problem. We also produce description of the relevant scattering data corresponding to the potential $v(x)$ from the Schwartz class.

## Integrable structure for the multipoint distribution of TASEP

Andrei Prokhorov

University of Michigan and St. Petersburg State University

Consider the totally asymmetric simple exclusion process (TASEP) on the line with the step initial condition. The associated Kolmogorov equation can be transformed to the dynamic equation for the XXZ quantum spin chain and can be solved using the coordinate Bethe ansatz. Its transition probabilities have explicit expressions, making this model a determinantal point process. The cumulative distribution function of the associated height function coincides with the cumulative distribution function of maximal eigenvalue of finite Laguerre Unitary Ensemble (LUE) of random matrices and it is expressed in terms of the solution of Painlevé V equation.

The infinite time scaling limit of this model belongs to the Kardar-Parisi-Zhang (KPZ) universality class. Denote its random space-time height function as $h(t, y)$. The one point distribution

$\text{Prob}(h(t,y) < x)$ as the function of three variables is the tau function the Kadomtsev-Petviashvili (KP) equation. The multipoint distribution $\text{Prob}(\cap_{k=1}^{n}\{h(t_k, y_k) < x_k\})$ have the representation as the multiple contour integral of the integrable Fredholm determinant. We show that this determinant is the tau function for the multicomponent KP hierarchy.

We will also review the various different random growth models, their scaling limits and the differential equations behind them.

This talk is based on the joint work with Jinho Baik and Guilherme Silva.

## Recent results on Electric Impedance Tomography of 2-dim Riemannian manifolds

Mikhail Belishev and Dimitrii Korikov

St. Petersburg Department of Steklov Mathematical Institute

Let $(M, g)$ be a smooth compact two-dimensional Riemannian manifold (*surface*) with a smooth metric tensor $g$ and smooth boundary $\Gamma$. Its *DN-map* $\Lambda : C^\infty(\Gamma) \to C^\infty(\Gamma)$ is associated with the (forward) elliptic problem $\Delta_g u = 0$ in $M \setminus \Gamma$, $u = f$ on $\Gamma$, and acts by $\Lambda f := \partial_\nu u^f$ on $\Gamma$, where $\Delta_g$ is the Beltrami-Laplace operator, $u = u^f(x)$ the solution, $\nu$ the outward normal to $\Gamma$. The corresponding *inverse problem* (EIT-problem) is to determine the surface $(M, g)$ from its DN-map $\Lambda$.

An algebraic version of the Boundary Control method (Belishev' 2003) is developed:

1) the version is extended to the case of *nonorientable* surfaces and a criterion of orientability (in terms of $\Lambda$) is obtained;

2) a characteristic description of $\Lambda$ that provides the necessary and sufficient conditions for solvability of the inverse problem for orientable surfaces, is given;

3) a procedure that determines surfaces with (unknown) internal holes, is proposed.

[1] M. I. Belishev, D. V. Korikov. On the EIT problem for nonorientable surfaces. Journal of Inverse and Ill-posed Problems; 18 December 2020. doi:10.1515/jiip-2020-0129.
[2] M. I. Belishev, D. V. Korikov. On characterization of Dirichlet-to-Neumann map of Riemannian surface with boundary. arXiv:2103.03944v1 [math.AP] 5 Mar 2021.

## Asymptotic analysis for step-like problems for integrable equations

Alexander Minakov

Charles University, Czech Republic

I will talk about asymptotic analysis of solutions of integrable equations with step-like data, in particular for the modified Korteweg-de Vries equation and the nonlinear Schrödinger equation. A particular attention will be paid to a transition zone between the dispersive shock wave region and the solitonic region, where the solution is described by a train of the so-called Khruslov's solitons.

# Birman conference

## Homogenization of Steklov sieve

Andrey Piatnitski

The Arctic University of Norway and Institute for Information Transmission Problems of RAS

The talk will focus on homogenization of a Steklov type spectral problem stated in a bounded domain with a thin periodically punctured interface, the Steklov boundary condition being imposed on the perforation surface. For an appropriate range of parameters we construct the effective spectral problem and justify the convergence of eigenpairs. We show in particular that the Steklov type boundary condition in the original problem is transformed to an interface spectral condition in the effective problem.

## Homogenization of the periodic Schrödinger-type equations

Tatiana Suslina

St. Petersburg State University

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a selfadjoint matrix strongly elliptic second order differential operator $A_\varepsilon$. It is assumed that the coefficients of $A_\varepsilon$ are periodic and depend on $\mathbf{x}/\varepsilon$, where $\varepsilon > 0$ is the small parameter. We study the behavior of the operator exponential $e^{-iA_\varepsilon\tau}$ for small $\varepsilon$ and $\tau \in \mathbb{R}$. The results are applied to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the Schrödinger-type equation $i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},\tau) = (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x},\tau)$ with the initial data from a special class. For a fixed $\tau \in \mathbb{R}$, the solution converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution of the homogenized problem, as $\varepsilon \to 0$; the error is $O(\varepsilon)$. We find approximation of the solution $\mathbf{u}_\varepsilon(\cdot,\tau)$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error $O(\varepsilon^2)$ and also approximation of $\mathbf{u}_\varepsilon(\cdot,\tau)$ in $H^1(\mathbb{R}^d;\mathbb{C}^n)$ with an error $O(\varepsilon)$. These approximations involve some correctors.

The research was supported by the Russian Science Foundation, grant no. 17-11-01069.

## Threshold resonances and virtual levels in the spectrum of periodic waveguides

Sergei A. Nazarov

St. Petersburg State University

A new classification of threshold resonances in the spectra of formally self-adjoint elliptic boundary value problems in cylindrical and periodic waveguides will be given. The main attention will be given to so-called degenerate thresholds, at which waves with polynomial growth at infinity may be considered as standing because they do not drive energy along the waveguide. Such waves can produce novel threshold anomalies which are surely absent in acoustic and quantum cylindrical waveguides (second-order differential equations). In particular, it will be demonstrated that an embedded eigenvalue may grow up from a degenerate threshold.

This work was financially supported by Russian Science Foundation (project 17–11–01003).

[1] Nazarov S. A., *Threshold resonances and virtual levels in the spectrum of cylindrical and periodic waveguides*, Math. Izvestiya, 84 (2020), no. 6, pp. 1105–1180.
[2] Nazarov S. A., *Constructing a trapped mode at low frequencies in an elastic waveguide*, Funct. Anal. Appl., 54 (2020), no. 1, pp. 31–44.

## Uniform asymptotics for families of degenerating variational problems and applications to error estimates in homogenization

Valery P. Smyshlyaev

University College London, UK

We study the uniform asymptotics of solutions to abstract families of degenerating variational problems, generalizing those emerging in the course of deriving operator estimates in periodic homogenization via Floquet–Bloch transform in the approach of Birman and Suslina. A hierarchy of approximation results with error estimates is established under various assumptions, valid for wide classes of examples. As a result, we provide in particular approximations of the spectrum in terms of the spectrum of some effective operator which generalizes the two-scale homogenized operator for "double-porosity" type high-contrast models. An explicit description of this effective spectrum is provided and error estimates on the distance between the original and effective spectra are established. Our approach starts as the well-established spectral method in (classical) homogenization, but the techniques we develop do not rely on analytic perturbation theory and this allows us to readily consider a much wider class of degenerating problems including high-contrast homogenization problems. The obtained results are illustrated by various examples, ranging from periodic homogenization to problems with concentrated masses, imperfect interfaces and non-local differential-difference equations. Joint with Shane Cooper and Ilia Kamotski.

## On homogenization for locally periodic elliptic problems on a domain

Nikita N. Senik

St. Petersburg State University

Let $\Omega$ be a Lipschitz domain, and let $\mathcal{A}^\varepsilon = -\operatorname{div} A(x, x/\varepsilon)\nabla$ be a strongly elliptic operator on $\Omega$ with fairly general boundary conditions, including, in particular, the Dirichlet and Neumann boundary conditions, as well as mixed ones. We suppose that the parameter $\varepsilon$ is small and the function $A$ in the operator $\mathcal{A}^\varepsilon$ is Lipschitz in the first variable and periodic in the second, so its coefficients are locally periodic and rapidly oscillating. It is a classical result in homogenization theory that the resolvent $(\mathcal{A}^\varepsilon - \mu)^{-1}$ converges (in a certain sense) as $\varepsilon \to 0$. We are interested in approximations for $(\mathcal{A}^\varepsilon - \mu)^{-1}$ and $\nabla(\mathcal{A}^\varepsilon - \mu)^{-1}$ in the operator norm on $L_p$ for a suitable $p$. The rates of the approximations depend on regularity of the effective operator $\mathcal{A}^0$. We prove that if $(\mathcal{A}^0 - \mu)^{-1}$ is continuous from $L_p$ to the Besov space $B^{1+s}_{p,\infty}$ with $0 < s \le 1$, then the rates are, respectively, $\varepsilon^s$ and $\varepsilon^{s/p}$.

The research was supported by Russian Science Foundation grant no. 17-11-01069.

## Semiclassical eigenvalues for the Schrödinger operators on surfaces with conic points and with $\delta$-potentials

Andrei Shafarevich

Moscow State University

We study semiclassical asymptotics of eigenvalues for Laplace and Schrödinger operators on surfaces of revolution, containing special singularities. Namely, we discuss the cases, when the corresponding surface contains conic points as well as the cases of operators with $\delta$-potentials. We describe Lagrangian surfaces, corresponding to semiclassical eigenvalues and obtain the analogues of quantization conditions. We also describe the effect of the jump of the Maslov index while passing through the critical value of boundary coefficient, corresponding to the singularity.

## Spectral properties of ordinary differential operators generated by a first order system

Andrey Shkalikov

Moscow State University

We consider operators generated by differential expressions of the form

$$l(\mathbf{y}) = \mathbf{B}(x)\frac{d\mathbf{y}}{dx} + \mathbf{Q}(x)\mathbf{y}, \quad \mathbf{y} = \{y_1, y_2, \dots, y_n\}, \;\; x \in [a,b],$$

and boundary conditions

$$\mathbf{U}_0\mathbf{y}(a) + \mathbf{U}_1\mathbf{y}(b) = 0.$$

Here $\mathbf{U}_0$ and $\mathbf{U}_1$ are $n \times n$ matrices, $\mathbf{B} = \operatorname{diag}\{b_1(x), b_2(x), \dots, b_n(x)\}$, and it is assumed that $b_j^{-1}$ and the entries of the $n \times n$ matrix-function $Q$ are summable.

We modify the concept of regularity (it was originated in the works of G. Birkhoff, J. Tamarkin, and R. Langer) and prove that the eigenfunctions and associated functions of a regular operator form an unconditional basis in $L_2(a,b)$. Some other properties of regular operators will also be discussed.

## On bifurcations of internal thresholds in essential spectrum under small non-symmetric perturbations

Denis Borisov

Institute of Mathematics, Ufa Federal Research Center, RAS

Let $x' = (x_1, \dots, x_{d-1})$, $x = (x', x_d)$ be Cartesian coordinates in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$, where $d \geqslant 2$, and $\omega \subseteq \mathbb{R}^{d-1}$ be an arbitrary domain. The domain $\omega$ can be either bounded or unbounded. If the boundary of this domain is non-empty, we assume that it belongs to $C^2$. By $A_{ij} = A_{ij}(x')$, $A_j = A_j(x')$, $A_0 = A_0(x')$, $i,j = 1, \dots, d-1$, we denote real-valued functions defined on $\overline{\omega}$ and possessing the following smoothness: $A_{ij}, A_j \in C^1(\overline{\omega})$, $A_0 \in C(\overline{\omega})$. The functions $A_{ij}$ are supposed to satisfy a usual ellipticity condition.

On the domain $\Omega := \omega \times \mathbb{R}$ we consider the operator

$$\mathcal{H} = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} - \frac{\partial^2}{\partial x_d^2} + \mathrm{i} \sum_{j=1}^{d-1} \left( A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \quad \text{in} \quad \Omega$$

subject to Dirichlet or Robin condition:

$$u = 0 \quad \text{on} \quad \partial\Omega \qquad \text{or} \qquad \frac{\partial u}{\partial \boldsymbol{\nu}} - a(x')u = 0 \quad \text{on} \quad \partial\Omega,$$

where

$$\frac{\partial u}{\partial \boldsymbol{\nu}} := \sum_{i,j=1}^{d-1} A_{ij} \nu_i \frac{\partial u}{\partial x_j} - \mathrm{i} \sum_{j=1}^{d-1} A_j \nu_j u + \nu_d \frac{\partial u}{\partial x_d}, \tag{4}$$

$\nu = (\nu_1, \ldots, \nu_d)$ is the unit normal vector to $\partial\Omega$, and $a = a(x')$ is a real function continuous and uniformly bounded on $\partial\Omega$.

We assume that the lower part of the spectrum of the self-adjoint operator

$$\mathcal{H}' = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \mathrm{i} \sum_{j=1}^{d-1} \left( A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \quad \text{in} \quad \omega$$

subject to the same boundary condition as in (4) consists of discrete eigenvalues $\Lambda_1 \leqslant \Lambda_2 \leqslant \cdots \leqslant \Lambda_m < c_0$, while its essential spectrum is located above the point $c_0$.

In the work we consider a small non-symmetric perturbation of the operator $\mathcal{H}$ of form $\mathcal{H}_\varepsilon := \mathcal{H} + \varepsilon\mathcal{L}(\varepsilon)$, where $\mathcal{L}(\varepsilon)$ is some linear not necessarily symmetric operator bounded as acting from $W_2^2(\Omega, e^{-\vartheta|x_d|}dx)$ into $L_2(\Omega, e^{\vartheta|x_d|}dx)$, while $\vartheta$ is some fixed positive constant. The essential spectrum of the operator $\mathcal{H}_\varepsilon$ read as $\sigma_{ess}(\mathcal{H}_\varepsilon) = [\Lambda_1, +\infty)$, and the aforementioned numbers $\Lambda_j$ serve as thresholds in this spectrum. The main result of the work is a detailed description of the bifurcation of the thresholds $\Lambda_j$ into the eigenvalues and resonances of the operator $\mathcal{H}_\varepsilon$. We obtain sufficient conditions ensuring the existence of such eigenvalues and the resonances. In the case of existence, we find the leading terms in their asymptotic expansions. We also show that the total multiplicity of the emerging spectral objects can be up to twice more than the multiplicity of the threshold $\Lambda_j$ considered as an eigenvalue of the operator $\mathcal{H}'$.

The talk is based on a joint work with D. A. Zezyulin and M. Znojil [1].

The research is supported by a grant of Russian Science Foundation (project no. 20-11-19995).

[1] Borisov D. I., Zezyulin D. A., Znojil M., *Bifurcations of thresholds in essential spectra of elliptic operators under localized non-Hermitian perturbations*, Stud. Appl. Math., 146 (2021), no. 4, pp. 834–880.

## Mixed norm spaces of analytic functions as spaces of generalized fractional derivatives of functions in Hardy type spaces

Alexey Karapetyants

Southern Federal University

The aim of the talk is twofold. First, we present a new general approach to the definition of a class of mixed norm spaces of analytic functions $\mathcal{A}^{q;X}(\mathbb{D})$, $1 \leqslant q < \infty$, on the unit disc $\mathbb{D}$. We study a problem of boundedness of Bergman projection in this general setting. Second, we apply this general approach for the new concrete cases when $X$ is one of the following spaces: variable exponent Lebesgue space, Orlicz space, generalized Morrey space, Orlicz–Morrey space, Grand or Small space or some other generalizations. In general, such introduced spaces are the spaces of functions which are in a sense the generalized Hadamard type derivatives of analytic functions having $l^q$ summable Taylor coefficients.

## Spectral properties of a singular string equation: continuous spectrum and eigenvalues

Igor A. Sheipak

Lomonosov Moscow State University

We consider the equation of oscillation of a singular string with a weight from the Sobolev space with a negative index of smoothness. If weight is a $n$-term self-similar non-compact multiplier, the spectral problem for the string is equivalent to the spectral problem for the $(n-1)$-periodic Jacobian matrix. In the case of $n = 3$, a complete description of the spectrum of the problem is given, and a criterion for the appearance of an eigenvalue in the gap of the continuous spectrum is obtained.

In the general situation $n \geqslant 3$, it is shown that the spectrum consists of $n - 1$ segments of the continuous spectrum (some of which may touch) in the gaps between which there can be no more than $n - 2$ eigenvalues (no more than one in each interval).

The talk is based on a joint work with Evgeny Sharov (Lomonosov Moscow State University).

## On the rate of decrease at infinity of solutions to the Schrödinger equation in a half-cylinder

Nikolay Filonov

St. Petersburg Department of Steklov Mathematical Institute, St. Petersburg State University

We consider the equation

$$-\Delta u + Vu = 0$$

in the half-cylinder $[0, \infty) \times (0, 2\pi)^d$ with periodic boundary conditions on the side surface. A potential $V$ is assumed to be bounded. We are interested in the possible rate of decreasing of a non-trivial solution $u$ at the infinity. Clearly, a solution can decrease exponentially. If $d = 1$ or $d = 2$, a solution can not decrease faster; if

$$u(x, y) = O\left(e^{-Nx}\right) \quad \forall\ N,$$

then $u \equiv 0$. Here $x$ is the axial variable. For $d \geq 3$, we construct an example of non-trivial solution decreasing as $e^{-cx^{4/3}}$, and we show that it is optimal,

$$u(x, y) = O\Big(e^{-Nx^{4/3}}\Big) \quad \forall\ N \qquad \Longrightarrow \qquad u \equiv 0.$$

This is a joint work with S. Krymski.

The research was supported by Russian Science Foundation grant no. 17-11-01069.

## Ballistic transport for one-dimensional quasiperiodic Schrödinger operators

Ilya Kachkovskiy

Michigan State University

Let $H$ be a discrete Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V_n\psi(n).$$

Denote by $X$ the position operator:

$$(X\psi)(n) = n\psi(n),$$

and by $X(T)$ its Heisenberg evolution:

$$X(T) = e^{iTH}Xe^{-iTH}.$$

We say that the operator $H$ satisfies *strong ballistic transport* if the limit

$$Q := \lim_{T\to+\infty} \frac{1}{T}X(T), \tag{1}$$

exists in the strong operator topology, and the operator $Q$, known as asymptotic velocity, has trivial kernel.

In a weaker form, ballistic motion is expected from all operators with absolutely continuous spectrum (Guarneri–Combes–Last theorem). However, the “true” strong ballistic transport has only been known for periodic and limit-periodic operators (Asch–Knauf, Damanik–Lukic–Yessen, Fillman). In the present talk, we will consider quasiperiodic operators

$$(H_x\psi)(n) = \psi(n-1) + \psi(n+1) + v(x+n\alpha)\psi(n), \quad n \in \mathbb{Z},\ x \in \mathbb{T}^d, \tag{2}$$

in several regimes of absolutely continuous spectrum. Partial results on ballistic motion for these operators have been obtained (Kachkovskiy, Zhang, Zhang–Zhao), however, they fall short of establishing (1) in different ways. In the present work, joint with L. Ge, we establish ballistic transport for a large class of operators (2) as a consequence of smooth reducibility of Schrödinger cocycles. Combined with the duality approach, quantitative bounds on the conjugation matrices (in other words, regularity of quasiperiodic Bloch waves) provide the missing ingredient that allows to transfer convergence information from the dual operator to the operator (2), preserving pointwise bounds. The required quantitative reducibility assumptions hold for a large class of operators, and can also be stated in terms of strong dynamical localization for the dual operators, also known in many cases (Germinet–Jitomirskaya, Jitomirskaya–Krüger, Ge–You–Zhou). We also introduce a concept of local ballistic transport, which is weaker than existence of (1), but only requires minimal regularity of conjugating matrices and does not rely on quantitative results.

## Ground state for nonlocal Schrödinger operator and spatially inhomogeneous contact models

Elena Zhizhina

IITP RAS

The asymptotic behavior of stochastic infinite-particle systems in continuum can be studied in terms of evolution equations for correlation functions. For the stochastic contact model in the continuum the evolution equation for the first correlation function (so-called density of population) can be considered separately from equations for higher-order correlation functions. In this case we get the following evolution problem for $u \in C([0,\infty);\mathcal{E})$ associated with a nonlocal generator $L$:

$$\frac{\partial u}{\partial t} = Lu, \quad u = u(t,x), \quad x \in \mathbb{R}^d, t \geq 0, \quad u(0,x) = u_0(x) \geq 0,$$

in a proper functional space $\mathcal{E}$. As $\mathcal{E}$, we consider two spaces: $C_b(\mathbb{R}^d)$, the Banach space of bounded continuous functions on $\mathbb{R}^d$, and $L^2(\mathbb{R}^d)$. These spaces correspond to two different regimes in the contact model: systems with bounded densities and with essentially localized densities.

The operator $L$ has the following form:

$$Lu(x) = -m(x)u(x) + \int_{\mathbb{R}^d} a(x-y)u(y)dy,$$

where $a(x) \geq 0$, $a(-x) = a(x)$, $a \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is an even continuous bounded function such that $\int_{\mathbb{R}^d} a(x)dx = 1$. The function $a(x-y)$ is the dispersal kernel associated with birth rates in the contact model. The function $m(x)$ is related with mortality rates. We assume here that

$$m \in C_b(\mathbb{R}^d), \quad 0 \leq m(x) \leq 1, \quad m(x) \to 1, \ |x| \to \infty.$$

We consider local perturbations of the stationary regime, when $m(x)$ is an inhomogeneous in space non-negative function. We prove that local fluctuations of the mortality with respect to the critical value $m(x) \equiv 1$ can push the system away from the stationary regime. As a result of such local perturbations, we will observe exponentially increasing density of population everywhere in the space. The main question is the existence of a positive discrete spectrum of the operator $L$. We prove that the positive discrete spectrum appears in two cases: either there exists such a region of any (small) positive volume, where the fluctuation $V(x) = 1 - m(x)$ is equal to 1, or $V(x)$ is positive and less than 1 in a large enough region. Thus we observe for the nonlocal operator $L$ new effects different from those of the Schrödinger operators. We found that even in the high dimensional case some small (in the integral sense) perturbations imply existence of a positive discrete spectrum.

The talk is based on the results of the paper [1].

[1] Yu. Kondratiev, S. Molchanov, S. Pirogov, E. Zhizhina, *On ground state of some non local Schrödinger operator*, Applicable Analysis, 96 (2017), no. 8.

## The structure of the Dirichlet–Laplacian spectrum in the Fichera layers and crosses of arbitrary dimension

Alexander I. Nazarov

PDMI RAS and St. Petersburg State University

We describe the Dirichlet–Laplacian spectrum structure for the Fichera layers and crosses in any dimension $n \geq 3$. The application to the classical Brownian exit times problem in these domains is given.

The talk is based on the joint papers [1,2]. Author's work was partially supported by the joint RFBR–DFG grant 20-51-12004.

[1] F. L. Bakharev, A. I. Nazarov, *Existence of the discrete spectrum in the Fichera layers and crosses of arbitrary dimension*, J. Funct. Anal., 281 (2021), no. 4, paper N109071, pp. 1–19.
[2] M. A. Lifshits, A. I. Nazarov, *On Brownian exit times from perturbed multi-strips*, Stat. & Prob. Letters, 147 (2019), pp. 1–5.

## Virtual levels of operators in Banach spaces and application to Schrödinger operators

Andrew Comech

Texas A&M University, College Station, Texas and IITP, Moscow, Russia

The concept of virtual levels comes from the study of scattering of neutrons on protons (E. Wigner, E. Fermi). It formed as a mathematical concept in articles of M. Birman, B. Vainberg, D. Yafaev, B. Simon, A. Jensen and T. Kato, and then many others (mostly in the context of selfadjoint Schrödinger operators). Virtual levels admit several equivalent characterizations: *(1)* there are corresponding *virtual states* from a space *slightly weaker* than $L^2$; *(2)* there is no limiting absorption principle in their vicinity (e.g. no weights such that the "sandwiched" resolvent is uniformly bounded); *(3)* an arbitrarily small perturbation can produce an eigenvalue. We develop a general approach to virtual levels starting from the following definition:

**Definition.** Let $A$ be a closed operator in a Banach space $\mathbf{X}$. Assume that $\mathbf{E}$, $\mathbf{F}$ are Banach spaces with dense continuous embeddings $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ such that $A$ has a closable extension to an operator $\hat{A}$ in $\mathbf{F}$. Given a connected open set $\Omega \subset \mathbb{C} \setminus \sigma(A)$ with $\sigma_{\mathrm{ess}}(A) \cap \partial\Omega \neq \emptyset$, we say that $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ is a *point of the essential spectrum of rank* $r \geq 0$ *relative to* $(\Omega, \mathbf{E}, \mathbf{F})$ if $r$ is the smallest value for which there is an operator $B \in \mathcal{B}(\mathbf{F}, \mathbf{E})$ of rank $r$ such that $\Omega \cap \sigma(A+B) \cap \mathbb{D}_\delta(z_0) = \emptyset$ with some $\delta > 0$ and such that there exists the following limit in the weak operator topology of mappings $\mathbf{E} \to \mathbf{F}$:

$$(A+B-z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}} := \operatorname*{w-lim}_{z\to z_0,\, z\in\Omega} (A+B-zI)^{-1} : \mathbf{E} \to \mathbf{F}. \qquad (1)$$

If $r = 0$, then $z_0$ is called a *regular point of the essential spectrum relative to* $(\Omega, \mathbf{E}, \mathbf{F})$. If $r \geq 1$, then $z_0$ is called a *virtual level* of rank $r$ relative to $(\Omega, \mathbf{E}, \mathbf{F})$. The corresponding space of virtual states is $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I) := \left\{\Psi \in \operatorname{Range}(A+B-z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}};\ (\hat{A} - z_0 I)\Psi = 0\right\}$, with $B$ a "regularizing" operator for which the limit (1) exists. This space does not depend on the choice of $B$, it is of dimension $r$, and it contains $\ker(A - z_0 I)$.

We provide applications to Schrödinger operators with nonselfadjoint potentials and in any dimension and derive optimal estimates on the resolvent.

This is a joint work with Nabile Boussaid based on the preprint [BC21].

[BC21] N. Boussaïd and A. Comech, *Virtual levels and virtual states of linear operators in Banach spaces. Applications to Schrödinger operators*, arXiv:2101.11979 (2021).

## The set of zeros of the Riemann zeta function as the point spectrum of an operator

Vladimir Kapustin

St. Petersburg Department of the Steklov Mathematical Institute

A Sturm–Liouville operator on a semiaxis and a small perturbation of it will be constructed so that the point spectrum of the resulting operator coincides with the set of non-trivial zeros of the Riemann zeta function after a simple transformation of the complex plane.

# Short talks

## On multiplicity properties of higher eigenvalues of the $p$-Laplacian

Vladimir Bobkov

Institute of Mathematics UFRC RAS

Let $B$ be a ball in $\mathbb{R}^N$, $N \geq 2$. Consider the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } B, \\ \qquad u = 0 & \text{on } \partial B, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p > 1$. A sequence of eigenvalues of this problem can be obtained by means of the following minimax variational principle:

$$\lambda_k(p) := \inf_{\mathcal{A}\in\Gamma_k} \max_{u\in\mathcal{A}} \frac{\int_B |\nabla u|^p \, dx}{\int_B |u|^p \, dx}, \quad k \in \mathbb{N},$$

where $\Gamma_k$ is a family of symmetric and compact subsets of $W_0^{1,p}(B)$ with Krasnosel'skii genus greater than or equal to $k$.

In the linear case $p = 2$, it is well-known that $\lambda_2(2) = \cdots = \lambda_{N+1}(2) = \lambda_\ominus(2)$, where $\lambda_\ominus(2)$ denotes the eigenvalue of the Laplacian whose eigenfunction vanishes on an equatorial section of $B$. We are interested in the generalization of this fact to the nonlinear settings. One of our main results is the following chain of inequalities satisfied for any $p > 1$:

$$\lambda_2(p) \leq \cdots \leq \lambda_{N+1}(p) \leq \lambda_\ominus(p).$$

If $\lambda_2(p) = \lambda_\ominus(p)$, as it holds true for $p = 2$, this result implies that the variational (algebraic) multiplicity of the second eigenvalue is at least $N$.

Audoux, B., Bobkov, V., & Parini, E. (2018). On multiplicity of eigenvalues and symmetry of eigenfunctions of the $p$-Laplacian. Topological Methods in Nonlinear Analysis, 51(2), 565-582.

## Spectral asymptotics for a fourth-order differential operator with multipoint boundary conditions

Dmitry Polyakov

Southern Mathematical Institute of Vladikavkaz Scientific Center of RAS

We consider a non-self-adjoint differential operator of the fourth order with nonsmooth periodic coefficients on a finite interval. The domain of this operator is defined by the multipoint Dirichlet boundary conditions. We obtain high energy asymptotic formulas for the eigenvalues and formula for regularized trace.

## Asymptotics in the form of special functions of eigenfunctions of the operator $\nabla D(\mathbf{x})\nabla$ defined in a two-dimensional domain and degenerating on its boundary

Anna Tsvetkova

Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences.

We study a spectral problem

$$- < \nabla, D(\mathbf{x})\nabla\psi(\mathbf{x}) >= \lambda\psi(\mathbf{x})$$

in a bounded two-dimensional domain $\Omega$, where the function $D(\mathbf{x})$ is a smooth positive function inside the domain and $D(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. Such operators arise in problems of long waves in an ocean with a variable bottom, defined by the function $D$, trapped by coasts and islands with the boundary $\partial\Omega$. For asymptotic solutions of this problem as $\lambda \to \infty$ explicit formulas are given in the case when the function $D(\mathbf{x})$ has a special form that guarantees the complete integrability of the Hamiltonian system with the Hamiltonian $H = D(\mathbf{x})|\mathbf{p}|^2$.

The eigenfunctions under consideration are associated with analogs of the Liouville tori of integrable geodesic flows with the metric degenerating on $\partial\Omega$. The feature of this problem is that the momentum components of the trajectories are infinite on the boundary of the domain. As a result, in addition to standard caustics (inside the domain), nonstandard caustics, formed

by the domain boundary, appear. In the neighborhood of standard caustics the asymptotic eigenfunctions are expressed in the form of the Airy function and in the neighborhood of nonstandard caustics – in the form of the Bessel function of a compound argument.
The talk is based on the joint work with A.Yu. Anikin, S.Yu. Dobrokhotov, V.E. Nazaikinskii.

The work was supported by Russian Science Foundation (project no. 16-11-10282).

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Asymptotic eigenfunctions of the operator $\nabla D(x)\nabla$ defined in a two-dimensional domain and degenerating on its boundary and billiards with semi-rigid walls. Differential Equations, **55**:5, 644—657 (2019).

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Nonstandard Liouville tori and caustics in asymptotics in the form of Airy and Bessel functions for two-dimensional standing coastal waves. Algebra and Analysis, **33**:2, 5—34 (2021) [in russian].

## Asymptotic solutions for nonlinear equations of gas dynamics, describing smoothed discontinuities

Anna Allilueva

Ishlinsky Institute for Problems in Mechanics RAS

We study asymptotic solutions for nonlinear equations of gas dynamics having rapid jump in a small vicinity of a moving surface. We prove that there are three different types of such solutions, describing smoothed shock waves, tangential jumps and smoothed weak jumps respectively. For each type of solutions we describe the motion of the surface of the jump as well as evolution of the profile.

# Summer school

## Lectures

### Spectral theory of Jacobi operators and asymptotic behavior of orthogonal polynomials

Dimitri Yafaev

Université de Rennes and St. Petersburg State University

We are planning to emphasize and use an analogy between differential operators and Jacobi operators (difference operators). Here is the outline of the course:
1. Definition and basic properties of Jacobi operators. Spectral measure
2. Two definitions of orthogonal polynomials (using recurrence relations and using an orthogonality measure)
3. Classical polynomials, Jacobi, Hermite and Laguerre, and their main properties
4. Freud weights
5. Jacobi operators with stabilizing coefficients. Generalizations of the Jacobi polynomials
6. Jacobi operators with increasing coefficients. Generalizations of the Hermite and Laguerre polynomials
7. Universal relations in asymptotic formulas.

### Spectral flow and some applications

Vladimir Nazaikinskii

Ishlinsky Institute for Problems in Mechanics RAS

The following topics will be discussed in the mini-course: The classical definition of spectral flow of a family of self-adjoint elliptic operators due to Atiyah, Patodi, and Singer and its role in index theory of elliptic operators. More convenient purely analytical version of the definition. Generalization of the notion of spectral flow to a family of parameter-dependent elliptic operators and index formulas for elliptic operators on manifolds with singularities. Spectral flow, partial spectral flow, and their application to the description of the Aharonov–Bohm effect in graphene. The exposition will be as elementary as possible, and no special prerequisites beyond the standard university course of mathematics will be required.

### Stochastic homogenization of convolution type operators and convolution type energies

Andrey Piatnitski

The Arctic University of Norway, UiT, campus Narvik and IITP RAS

The goal of the course is to consider a number of homogenization problems for non-local convolution type symmetric operators in random statistically homogeneous environments, and to justify the corresponding homogenization results. We are also going to discuss homogenization of variational problems for integral convolution type functionals in random media.

In the first part a number of auxiliary results will be presented. Among them are Poincare type inequalities, compactness of the family of solutions, extension theorems, and some other statements. Then we are going to discuss the subadditive ergodic theorem.

The second part of the course will focus on homogenization theorems for a number of non-local problems in random media.

## A short introduction to the theory of ergodic operators

Alexander Fedotov

St. Petersburg State University

First I briefly describe basic results and tools of the spectral theory of ergodic (e.g., almost periodic and random) operators: theorems on the typical spectra, definitions and properties of the integrated density of states and Lyapounov exponent, a relation between them, characterization of the absolutely continuous spectrum in terms of the Lyapounov exponent, examples of operators with pure point, singular continuous and absolutely continuous spectra. The second part of my lectures is a brief introduction to the monodromization method, a renormalization approach that was suggested by Buslaev and Fedotov when trying to generalize ideas of the Bloch–Floquet theory to quasiperiodic operators.

## Eigenvalues and resonances emerging from thresholds in essential spectra

Denis Borisov

Institute of Mathematics, Ufa Federal Research Center, RAS

A classical fact in the perturbation theory for self-adjoint unbounded operators states that under a perturbation of a given operator ensuring the norm resolvent convergence, the spectrum of the perturbed operator converges to that of the limiting operator. Simultaneously, the convergence of the spectral projectors holds. At the same time, this does not mean that under the perturbation the type of the spectrum should be preserved. An example is given by a well-known effect that localized perturbations of self-adjoint differential operators can generate bifurcations of certain threshold in their essential spectrum, namely, eigenvalues and resonance can emerge from such thresholds. Such phenomenon was known to physicists for many years and the first rigorous study for localized Schrödinger operators was done in classical papers by B. Simon in 70s. After these pioneering works, this phenomenon was studied for many various models.

In the proposed mini-course we shall discuss the main ideas and techniques underlying the described bifurcations of the thresholds. Although originally the technique by B. Simon and others was developed for studying self-adjoint operators, in 2000–2010 a modification of their approach was suggested and this allowed the researchers to study general, not necessarily symmetric or self-adjoint perturbations. Exactly these techniques will be discussed within the mini-course. Moreover, recently it was found that in certain cases, localized perturbations can *increase* the total multiplicity, for instance, a threshold of a generalized multiplicity one can bifurcate in two simple eigenvalues or in a simple eigenvalue and a resonance. Surprisingly, such situation can occur even for symmetric perturbations of self-adjoint operators. Such models will also be considered within our mini-course.

### High-frequency scattering by boundary inflection: a model for asymptotic transition from discrete to continuous

Valery Smyshlyaev
University College London

The problem of whispering gallery wave scattering by boundary inflection is a long-standing one in high-frequency diffraction. In 1976 M. M. Popov reduced it to an “inner” problem described by Schrödinger equation on a half-line with a potential linear in both space and time. The latter problem is arguably as fundamental for describing transition from a “modal” to a “scattered” high-frequency asymptotic behaviors, as Airy ODEs are for transition from oscillatory to exponentially decaying patterns. The sought solutions have asymptotic behaviors with a discrete spectrum at one end and with a continuous spectrum at the other end, and of central interest is to describe the map connecting these two asymptotic regimes.

We review the background, and then discuss a recent result proving that the solution past the inflection point has a “searchlight” asymptotics corresponding to a beam concentrated near the limit ray. This is achieved by a non-standard perturbation analysis at the continuous spectrum end, and we then discuss interpretations in terms of associated wave operators and of a scattering operator connecting the modal and the scattered asymptotic regimes. We also discuss some most recent progress on a reduction of the inner problem to one-dimensional boundary integral equations and their further analysis. The integral equations are of improper weakly singular Volterra type of both first and second kinds, and can be shown to be well-posed. Their subsequent regularization allows to express the solution in term of limit of uniformly convergent Neumann series with anticipated further benefits for the problem’s asymptotic analysis. We finally discuss the related open problems. Some parts of this course are based on joint recent works with Ilia Kamotski and Shiza Naqvi.

## Short talks

### Asymptotics of fundamental solutions to $2 \times 2$ first order system of ordinary differential equations

Alexei Kosarev
Moscow State University

We deal with a system of differential equations of the form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \lambda \begin{pmatrix} g & 0 \\ 0 & -h \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1}$$

where all involved functions are summable, $g, h > 0$, $p, q, r, s$ are complex-valued and $\lambda$ is the complex (spectral) parameter. This equation includes eigenvalue problems for the Dirac operator ($g = h = 1$) and for the system of telegraph equations ($g = h$).

Our objective is to obtain asymptotic behavior of fundamental solutions to the system (1) assuming minimal requirments on the smoothness of the involved functions. In particular, we find that there exists a fundamental matrix of solutions to (1) of the form

$$\begin{pmatrix} \exp\{\lambda G(x)\} & 0 \\ 0 & \exp\{\lambda H(x)\} \end{pmatrix} \left[ \begin{pmatrix} P(x) & 0 \\ 0 & S(x) \end{pmatrix} + o(1) \right],$$

where $o(1) \to 0$ uniformly in $C$-norm as $\lambda \to \infty$ in any fixed complex half-plane containing the imaginary axis.

In the case $g, h, p, q, r, s \in W_1^k[0,1]$ we find $k+1$ terms of the asymptotic expansion with an accuracy up to $o(\lambda^{-k})$ and write down explicit recurrent formula for the representation of the expansion coefficients.

The asymptotic theory for differential equations and systems with parameter raised up in works of G. Birkhoff, J. Tamarkin and R. Langer in the beginning of the last century. Works of B. Mityagin and P. Jakov, A. Savchuk and A. Shkalikov, A. Lunyov, M. Malamud and L. Oridoroga in the last decade made this topic actual again.

The report is based on the joint work with A. A. Shkalikov.

## Embedding constants in Sobolev spaces

Tatiana Garmanova

Lomonosov Moscow State University

We consider the problem of finding the norm of embedding operator $J_{n,k} : \mathring{W}_2^n[0;1] \to \mathring{W}_\infty^k[0;1]$, $n > k \geqslant 0$. For any function $f(x)$ from the space $\mathring{W}_2^n[0;1]$, we study the functions $A_{n,k}(x)$, that lead to the best possible estimates in inequalities of the type

$$|f^{(k)}(x)| \leqslant A_{n,k}(x) \|f^{(n)}\|_{L_2[0;1]}.$$

Then $\Lambda_{n,k} := \|J_{n,k}\| = \max\limits_{[0,1]} |A_{n,k}(x)|$. The properties of the local maxima of the functions $A_{n,k}(x)$ were found by a generalization of the Sonin-Polya's theorem, and the points of the global maxima were determined. The explicit representation of the functions $A_{n,k}^2$ were obtained in terms of hypergeometric functions. We also found the exact values of the embedding constants $\Lambda_{n,k}$ for all even $k$ and two-sided estimates of $\Lambda_{n,k}$ for odd $k$. The symmetry of the extremal function is proved in the case of even $k$.

## Eta-invariant for parameter-dependent families with periodic coefficients

Konstantin Zhuikov

RUDN University

On a smooth closed manifold, we consider a family of operators of the form of an infinite sum of parameter-dependent pseudodifferential operators with periodic coefficients. For the noted class of families, we introduce the $\eta$-invariant (of Atiyah-Patodi-Singer type) as a regularized winding number. For this purpose, certain regularizations for the trace of the operator and the integral are introduced. Further, we establish main properties of the $\eta$-invariant and present a formula for variation of the $\eta$-invariant as the family changes.

## Gaussian beam solutions to the Cauchy problem for the Schrödinger equation with a delta potential

Olga Shchegortsova

Moscow Institute of Physics and Technology (National Research University)

We study semiclassical asymptotics with complex phases to the Cauchy problem for the Schrödinger equation with a delta potential localized on a surface $M$ of codimension 1. The solutions of this type are known as Gaussian wave beams. It is well known that asymptotics of solutions to a Schrödinger equation with a smooth potential are constructed with help of Maslov's complex germ theory. We base our approach on the Maslov's construction.

The Schrödinger operator with a delta potential is defined as a self-adjoint extension of the Schrödinger operator with a smooth potential restricted to the functions vanishing on $M$ and requires boundary conditions for the solution on the surface $M$.

In the present work we present a construction of the gaussian beams satisfying the boundary conditions induced by the domain of the Schrödinger operator with a delta potential and describe the reflection of a Lagrangian manifold with complex germ.

## On adiabatic evolution generated by a one-dimensional Schrödinger operator

Vasily Sergeev

St. Petersburg State University

We consider a Schrödinger operator on the half-line with a potential slowly depending on time (namely, a slowly shrinking potential well). The spectrum of this operator consists of an absolutely continuous part $[0, +\infty)$ and a finite number of negative eigenvalues. With time the eigenvalues approach the edge of the continuous spectrum and, having reached it, disappear one by one. We study a solution of the corresponding non-stationary Schrödinger equation close at some point to an eigenfunction of the stationary operator. While the associated eigenvalue exists, the quantum particle described by this solution is localised within the potential well. We describe the delocalisation, which occurs when the eigenvalue disappears.