Quasiperiodic operators with monotone potentials

I. Kachkovskiy, MSU

November 30, 2020

Euler Institute, St. Petersburg
(online)

Lecture 2: 1d monotone potentials
(plus the remainder from Lecture 1)

https://users.math.msu.edu/users/ikachkov/minicourse.html

$$(H(x)\psi)_n = \psi_{n+1} + \psi_{n-1} + f(x+n\alpha)\psi_n \qquad \ell^2(\mathbb{Z})$$

$$H_n(x) = \begin{pmatrix} f(x) & 1 & & & \\ 1 & f(x+\alpha) & 1 & & 0 \\ & 1 & f(x+2\alpha) & & \\ & & & \ddots & \\ & & & & 1 \\ & 0 & & 1 & f(x+(n-1)\alpha) \end{pmatrix}$$

Let $N_n(x,E) = \# \sigma(H_n(x)) \cap (-\infty, E]$ (counting function).

$$N(E) := \lim_{N\to\infty} \frac{1}{N} \int_0^1 N_n(x,E)\,dx$$

integrated density of states = distribution function of the density of states measure.

$$dN = \int_0^1 \langle dE_{H(x)} e_0, e_0 \rangle dx :$$ spectral measure averaged in x

$$M_n(x,E) = \begin{pmatrix} P_n(x,E) & -P_{n-1}(x+\alpha,E) \\ P_{n-1}(x,E) & -P_{n-2}(x+\alpha,E) \end{pmatrix}$$ n-step transfer matrix

$$H(x)\psi = E\psi$$

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln \| M_n(x,E) \| dx = \inf_n \frac{1}{n} \int_0^1 \ln \| M_n(x,E) \| dx$$

Thouless formula:

$$L(E) = \int_{\mathbb{R}} \ln|E-E'|\,dN(E')$$

Heuristics: $L(E)$ "$\approx$" $\frac{1}{n}\int_0^1 \ln|P_n(x,E)|\,dx =$

$$= \frac{1}{n}\int_0^1 \sum_{j=0}^{n-1} \ln|E-E_j(x)|\,dx \approx \int_0^1 \ln|E-E'|\,\underbrace{N_n}_{\text{counting measure}}(dE')$$

This can be made rigorous by considering $E \in \mathbb{C}^+$ and showing that L is subharmonic on $\mathbb{C}^+$.

Let $(H\psi)_n = \psi_{n+1} + \psi_{n-1} + V_n\psi_n$ (general operator)

We say that ψ is a <u>generalized eigenfunction</u> if

- $|\psi(n)| \le C_1(1+|n|)^{C_2}$ (polynomially bounded)
- $H\psi = E\psi$ for some $\underbrace{E}_{\text{generalized eigenvalue.}} \in \mathbb{R}$

In general, $\psi \notin \ell^2(\mathbb{Z})$.

<u>Theorem (Snol')</u> The spectral measure of H is supported on the set of generalized eigenvalues of H.

Many modern proofs of AL use Green's function method.

Let $H_N = H|_{[0,N-1]}$, $H\Psi = E\Psi$.

Then $H_N \Psi = E\Psi - \Psi_{-1} e_0 - \Psi_N e_N$

$$\Psi_m = -(H_N - E)^{-1}_{0m} \Psi_{-1} - (H_N - E)^{-1}_{m,N-1} \Psi_N$$

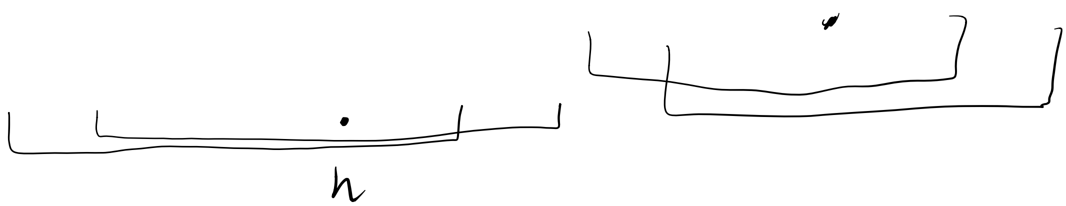
Then, try to prove that $(H_N - E)^{-1}$ has exponential decay away from the diagonal.

Cramer's rule:

$$(H_N - E)^{-1}_{m,N-1} = \frac{P_m(x,E)}{P_N(x,E)} \overset{?}{\sim} e^{(m-N)L(E)}$$

For most x, $P_m(x,E) \sim e^{mL(E)}$, $P_N(x,E) \sim e^{NL(E)}$

In both cases, "$\lesssim$". Therefore, need to avoid points where $P_N(x,E) \ll e^{NL(E)}$.



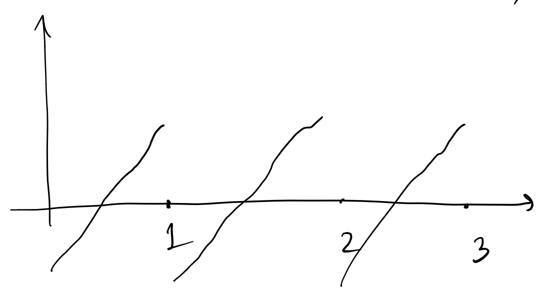
Some freedom in choosing the Box! [Jitomirskaya, 1998]

Relies on $L(E)>0$.

$$(H(x)\,\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + f(x+n\alpha)\,\Psi_n$$

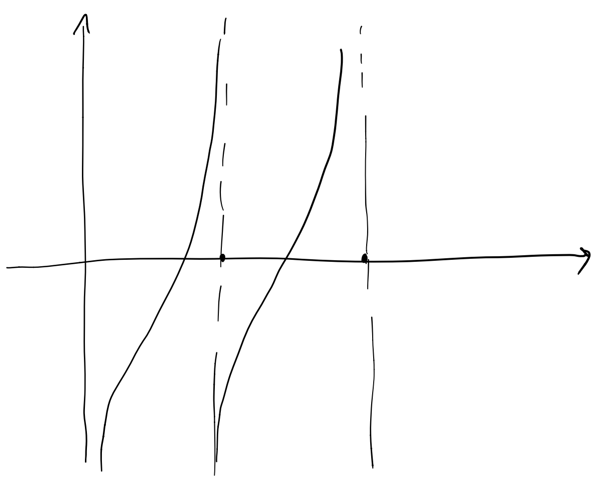
$f(x)$ is Lipschitz monotone on $(0,1)$:

$$f(y) - f(x) \geq \gamma\,(y-x), \qquad \gamma > 0, \qquad 0 < x < y < 1.$$



$f(x+1) = f(x)$

$\{x\}$ - type (bounded)



Maryland - type (unbounded)

Example: $f(x) = \tan(\pi x)$

In both cases:

1) $L(E) > 0$ for Lebesgue a.e. E

2) $L(E) \geq \min\{\ln(\gamma/2e), 0\}$.

$$H_n(x) = \begin{pmatrix} f(x) & 1 & & & \\ 1 & f(x+\alpha) & 1 & & 0 \\ & 1 & f(x+2\alpha) & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & & & 1 & f(x+(n-1)\alpha) \end{pmatrix}$$

$x \mapsto x + m\alpha$

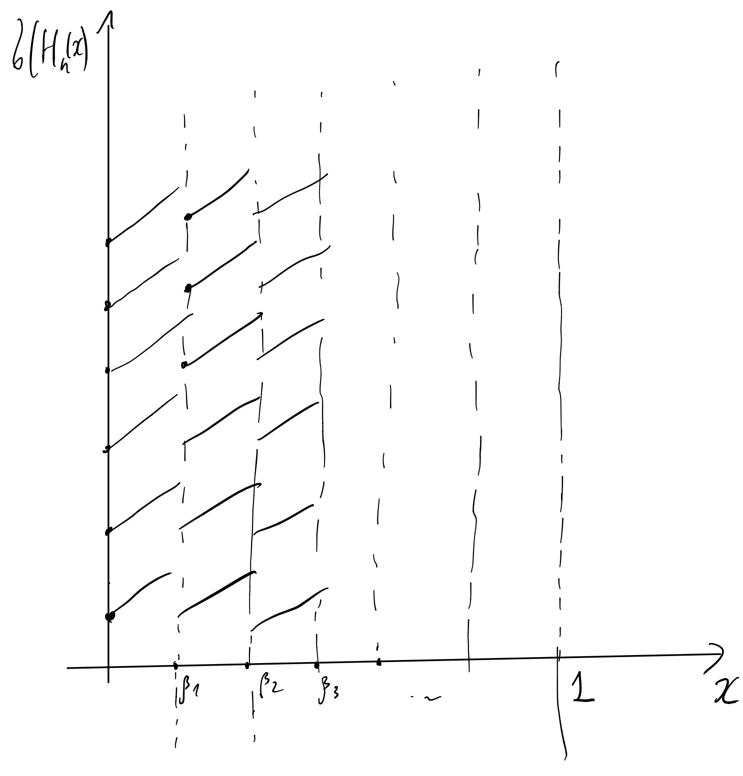
$[m, m+n]$

Th] In both cases, $|N(E_1)-N(E_2)| \le \gamma^{-1}|E_1-E_2|$

IDS

Recall: need to look at counting functions:

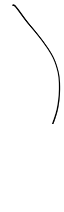
$$\frac{1}{n}\int_0^1 N_n(x,E)\,dx, \qquad N_n(x,E) = \#\sigma(H_n(x)) \cap (-\infty, E]$$

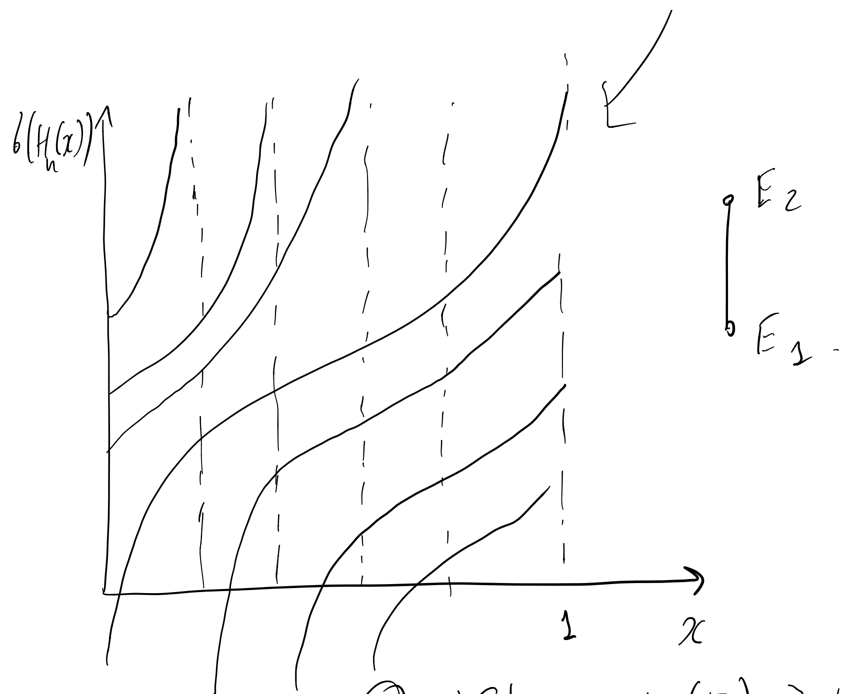


$$\begin{pmatrix} f(x) & 1 & & & \\ 1 & f(x+\alpha) & 1 & & 0 \\ & 1 & f(x+2\alpha) & 1 & \\ & & 1 & \ddots & \ddots \\ & 0 & & \ddots & 1 \\ & & & 1 & f(x+(n-1)\alpha) \end{pmatrix}$$

$\{x\}$-type case:

Maryland-type case:





Exercises: 1) a) Show: $L(E) \geq \min\{\ln(\gamma/2e), 0\}$ using Thouless formula

b) Show that $L(E)$ is continuous in E

2) Explain Maryland-type picture (why all eigenvalues except one are continuous)

3) Let $\alpha = p/q$, and

$$\tilde{H}_n(x) = \begin{pmatrix} f(x) & 1 & & & 1 \\ 1 & f(x+\alpha) & 1 & 0 & \\ & 1 & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1 \\ 1 & & & 1 & f(x+(n-1)\alpha) \end{pmatrix}$$

Any additional structure in the picture?

Let $Z = \{E : L(E) = 0\}$

$|Z| = 0 \Longrightarrow N(Z) = 0 \Longrightarrow$

density of states measure

$$\Longrightarrow \int_0^1 \langle \mathbb{E}_{H(x)}(Z) e_j, e_j \rangle \, dx = 0.$$

Therefore, for a.e. x, Z has zero spectral measure!

Now we can restrict ourselves to $H(x)\psi = E\psi$, $L(E) = 0$.

Let $P_n(x, E) = \det(H_n(x) - E) = \prod_{j=0}^{n-1} (E_j(x) - E)$

The "jump points" on the above picture are at $\{-j\alpha\}$, $j = 0, 1, \ldots, n-1$.

Let $\frac{p_j}{q_j} \to \alpha$ (continued fractions), and let q be one of the q_j. We will consider $n=q$.

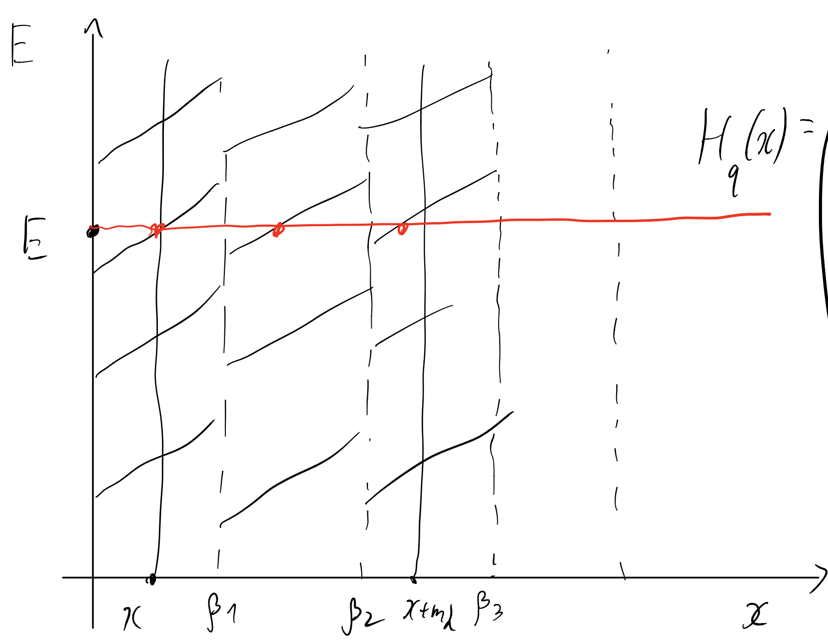
The jump points are distributed regularly:

$$\beta_0 \quad \beta_1 \quad \beta_2 \quad \dots \quad \beta_{q-1}$$

$\frac{1}{2q} \le |\beta_{j+1} - \beta_j| \le \frac{3}{2q}$. Each interval $\left(\frac{j}{q}, \frac{j+1}{q}\right)$ contains one β_j

$$H_q(x) \approx H_q(x+m\alpha)$$

counting functions differ by a constant



$$H_q(x) = \begin{pmatrix} f(x) & 1 & & 0 \\ 1 & f(x+\alpha) & 1 & \\ & 1 & \ddots & 1 \\ 0 & & 1 & f(x+(q-1)\alpha) \end{pmatrix}$$

$x \mapsto x + m\alpha$.

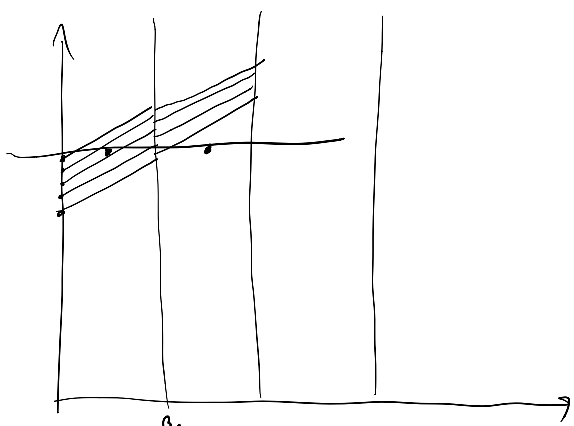
$\{q\alpha\}$

$$P_n(x,E) = \det(H_n(x) - E)$$

Ultimately, $|N_q(x,E) - N_q(y,E)| \le C$

Separation of eigenvalues modulo finite rank is as good as γ. (separation of $\{x+j\alpha\}$)

$j = 0, 1, \dots, q-1$



Large deviation set: q intervals of size $O(e^{-cq})$

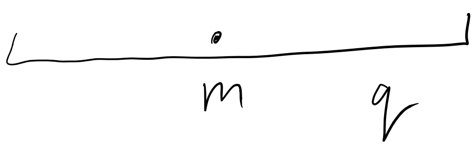
Call $m \in \mathbb{Z}$ q-singular if one cannot find a good q-interval around m.

WLOG, 0 is q-singular for large q.

Theresore, $[q, e^{cq}]$ cannot contain singular points.

$$\Psi_m = -(H_N - E)^{-1}_{0m}\,\Psi_{-1} - (H_N - E)^{-1}_{m,N-1}\,\Psi_N$$

$$\frac{P_m(x,E)}{P_N(x,E)}$$



q-regular

$$|m - m'| \geq \frac{q+1}{2}$$

Both singular.

$P_N(x + m;\alpha)$ is small

$2q(1 - o(1))$

$m;\alpha$ in large deviation set

union of q intervals.

$$\operatorname{dist}(m_j\alpha - m_k\alpha, \mathbb{Z}) < e^{-cq}$$

$$\vee\!/$$

$$c\,|m_j - m_k|^{-\tau}$$

$$|m_j - m_k| \gg e^{c_1 q}$$

Theorem: Let f be γ-monotone

α be Diophantine. $\|n\alpha\| \geq c|n|^{-\tau}$

Then $H(x)$ has localization:

1) for a.e. x

2) for all x if $\gamma > 2e$

Appendix 1:

Let V be an ergodic potential. We say that V is <u>deterministic</u> if $V|_{\mathbb{Z}_-}$ determines V.

Example: random potentials are not deterministic.

<u>Th</u> (Kotani). Let V be non-deterministic. Then $L(E) > 0$ for Lebesgue a.e. E.

$\{x\}$ seems deterministic. However, one can consider a "closure":

$$\begin{array}{ccc} \mathbb{T}^1 & \longrightarrow & [a,b]^n \\ \Psi & & \Psi \\ x & \longmapsto & \{\{x+n\omega\} : n \in \mathbb{Z}\} \\ d\theta & \longmapsto & d\partial . \end{array}$$

(Haar measure)

([a,b] is the range of f)

The "extended" ergodic potential is not deterministic on $\operatorname{supp} d\partial$

(Damanik, Killip).

Unbounded case:

<u>Th (Simon, Spencer)</u> Let V be unbounded. Then $\sigma_{ac}(H+V) = \emptyset$.

Therefore, in either case $L(E)>0$ for a.e. E (Lebesgue),

and we do not need to worry about σ_{ac}.

However, the set $Z=\{E: L(E)=0\}$ can still have positive spectral measure.

Appendix 2:

<u>Th (Furman)</u> Let (Ω, μ, T) be a uniquely ergodic system,

$f: \Omega \to [-\infty, +\infty)$ be subadditive:

$$f_{n+m}(x) \le f_n(x) + f_m(T^n x),$$

$$f \in L^1(X, \mu).$$

Let $L(f) = (\text{a.e.}) \lim_{n\to\infty} \frac{1}{n} f_n(x)$.

Then, uniformly on $x \in X$,

$$\limsup_{n\to\infty} \frac{1}{n} f_n(x) \le L(f).$$

For $\{x\}$-type, can apply directly.

For Maryland-type, "factor out" the unbounded part and analyze separately.

Appendix 3: Originally, this method was developed for the almost Mathieu operator:

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + 2\lambda \cos 2\pi(x+n\alpha)\,\psi_n.$$

In this case, P is a polynomial in $\cos x$.

n roots $\longrightarrow$ $2n$ roots $\longrightarrow$ resonance.

In monotone case, no such resonances!

Leads to uniform localization

$$\psi_n^{(j)} \leq C(\delta)\, e^{-(L(E)-\delta)|n-n_j|}\, \psi_n^{(j)}(n_j)$$