Quasiperiodic operators with monotone potentials

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Lecture 3: Perturbative methods in higher dimensions

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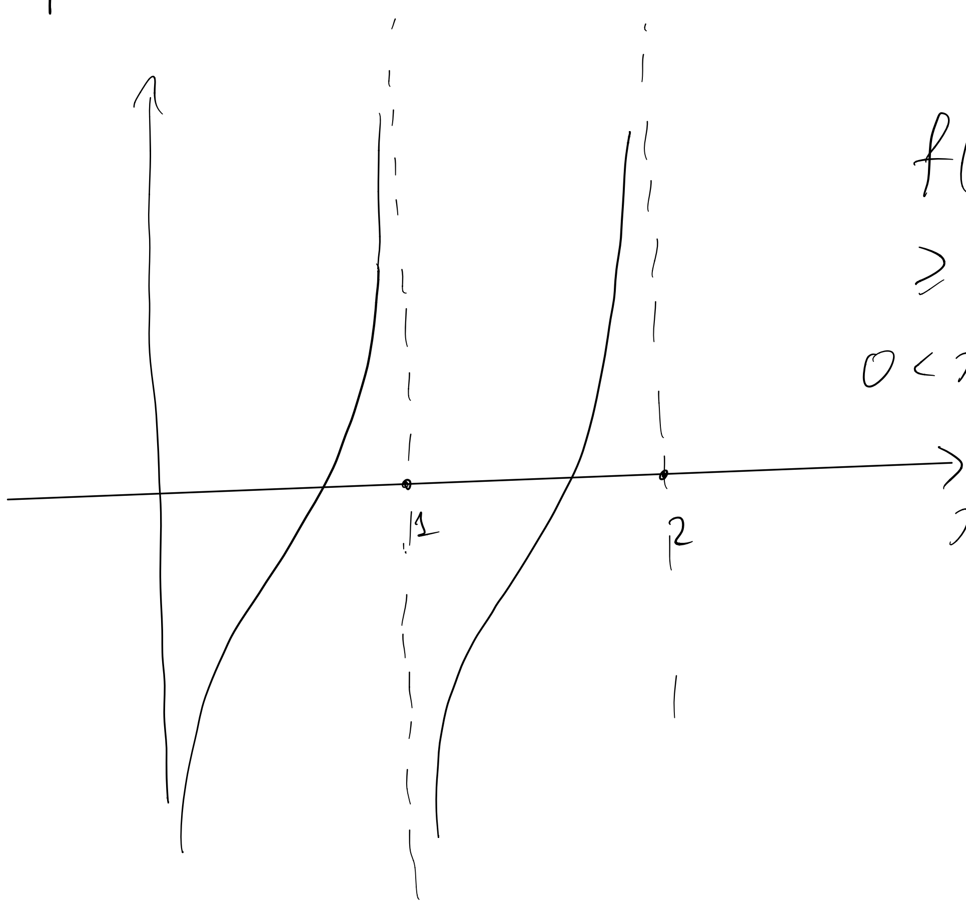
On $\ell^2(\mathbb{Z}^d)$, consider

$$(H(x)\Psi)_n = \varepsilon(\Delta\Psi)_n + f(x + n\cdot\alpha)\Psi_n$$

$\{1, \alpha_1, \ldots, \alpha_d\}$ independent over $\mathbb{Z}$.

f is Maryland-type:



$f(y) - f(x) \geq$
$\geq \gamma(y-x)$
$0 < x < y < 1$

$\gamma > 0$.

$\alpha \in DC$: $\operatorname{dist}(n\cdot\alpha, \mathbb{Z}) \geq C|n|^{-\tau}$.

Maryland model, $f(x) = \tan(\pi x)$.

Localization for all $\varepsilon > 0$, all d.

(Grempel, Fishman, Prange, Simon, Figotin, Pastur, ... see references)

General monotone analytic f:

Bellissard-Lima-Scoppola, $\varepsilon << 1$.

Proof is based on KAM-type iterations.

Goal: describe a more "naive" approach, which still works!

Rayleigh-Schrödinger perturbation series

For simplicity, consider $\ell^2(\mathbb{Z})$.
Extension to $\ell^2(\mathbb{Z}^d)$ is straightforward.

$$H = V + \varepsilon\Delta \qquad (V\psi)_n = V_n \psi_n .$$

Algebraic non-resonant condition: $V_n \neq V_m$ for $n \neq m$

$$\varepsilon = 0 : \quad H e_n = V_n e_n$$

Kato book

pretend that V_n are isolated, and write down the formal perturbation series.

WLOG: $n=0$, $V_0 = 0$ $\quad e_0$

$$H\psi = E\psi$$

$$(V + \varepsilon\Delta)(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots) = (E_0 + \varepsilon E_1 + \dots)(\psi_0 + \varepsilon\psi_1 + \dots).$$

Normalization: $\psi_0 = e_0$, $E_0 = 0$, $\langle \psi_j, e_0 \rangle = \delta_{j0}$.

$$V^{-1} e_j = \begin{cases} 0, & j=0 \\ V_j^{-1} e_j, & j=0 \end{cases}$$

"Moore-Penrose inverse" of V.

$$(V + \varepsilon\Delta)(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots) = (E_0 + \varepsilon E_1 + \dots)(\psi_0 + \varepsilon\psi_1 + \dots).$$

Solving the equations

$$(\Delta\Psi)_n = \sum_{|m-n|_1=1} \Psi_m$$

ε^0 $\quad \Psi_0 = e_0, \quad E_0 = 0$

ε^1 $\quad \Delta \Psi_0 + V\Psi_1 = E_1 \Psi_0 + \cancel{E_0}\, \Psi_1$

$$\Psi_j \perp e_0$$
$$j \neq 0.$$

$\langle e_0, \cdot \rangle$: $\quad E_1 = 0$

$$V\Psi_1 = -\Delta \Psi_0$$

Project onto $e_0^\perp$:

$$\Psi_1 = -V^{-1}(\Delta \Psi_0) = \frac{1}{-V_{-1}} e_{-1} + \frac{1}{-V_1} e_1$$

In general, $-V_j \longrightarrow V_0 - V_j$

ε^2 $\quad \Delta \Psi_1 + V\Psi_2 = E_2 \Psi_0 + \cancel{E_1}\,\Psi_1 + \cancel{E_0}\,\Psi_2$

$\langle e_0, \cdot \rangle$: $\quad E_2 = \langle e_0, \Delta \Psi_1 \rangle = -\frac{1}{V_{-1}} - \frac{1}{V_1}$

$e_0^\perp$: $\quad \Psi_2 = -V^{-1}(\Delta \Psi_1) =$

$$= \frac{1}{(-V_{-2})(-V_{-1})} e_{-2} + \frac{1}{(-V_1)(-V_2)} e_2$$

$$E_0, \Psi_0 \to E_1 \to \Psi_1 \to E_2 \to \Psi_2 \to \dots$$

$$(V+\varepsilon\Delta)(\Psi_0+\varepsilon\Psi_1+\varepsilon^2\Psi_2+\dots) = (E_0+\varepsilon E_1+\dots)(\Psi_0+\varepsilon\Psi_1+\dots)$$

$$E_0=0, \quad E_{2j+1}=0$$

ε^n: $$V\Psi_n + \Delta\Psi_{n-1} = \cancel{E_0}\Psi_n + \cancel{E_1}\Psi_{n-1} + E_2\Psi_{n-2}+\dots+E_{n-1}\Psi_1 + E_n\Psi_0$$

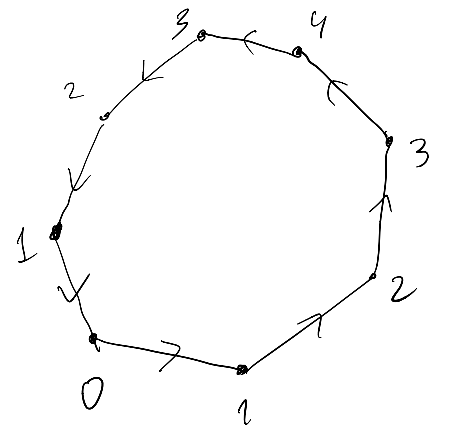
$\langle e_0,\cdot\rangle$: $$E_n = \langle e_0, \Delta\Psi_{n-1}\rangle$$

$e_0^\perp$: $$\Psi_n = -V^{-1}\Delta\Psi_{n-1} + \underbrace{V^{-1}E_2\Psi_{n-2}+V^{-1}E_4\Psi_{n-4}+\dots}_{\text{"attachment terms"}}$$

"simple loop term"

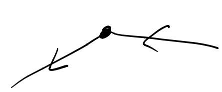
Simple loop:

$$\Psi_n = -V^{-1}\Delta\Psi_{n-1} = -V^{-1}\Delta(-V^{-1})\Delta\Psi_{n-2}\dots$$



Vertex factor:

n $\quad (-V_n)^{-1}$



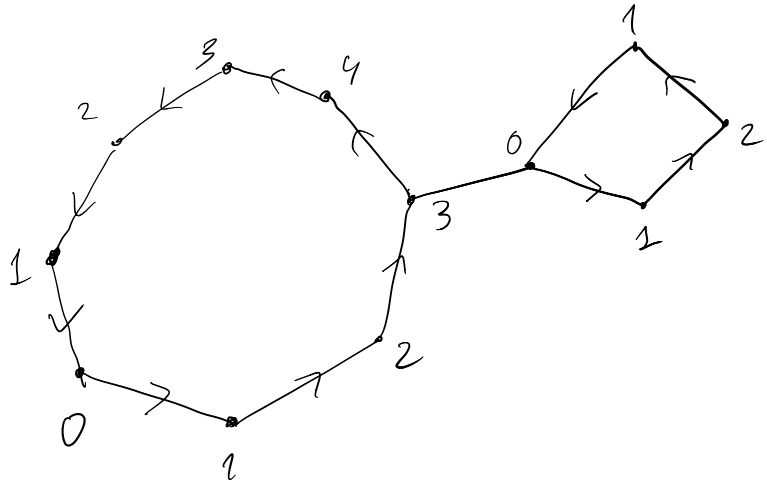
Edge factor

$\leftarrow \qquad \varepsilon$

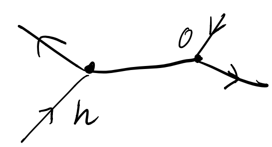
$$\frac{\varepsilon^8}{(-V_1)^2(-V_2)^2(-V_3)^2(-V_4)}$$

Suppose that we are using attachment term <u>once</u>

We get a product of two simple loops, with an extra $(V_j)^{-1}$ factor.



Attachment factor:



V_n^{-1}

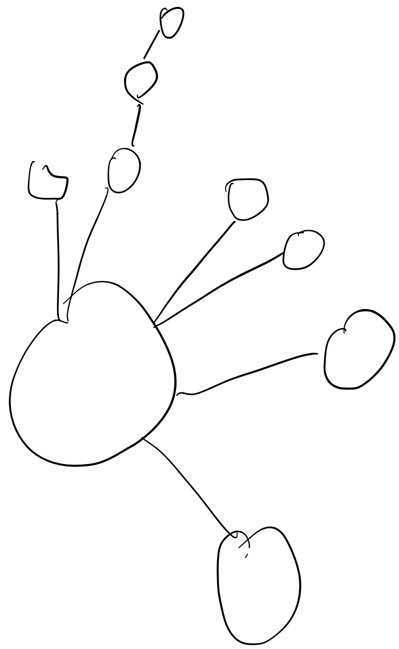
$$\frac{\varepsilon^8}{(-V_1)^2(-V_2)^2(-V_3)^2(-V_4)} \cdot \frac{1}{V_3} \cdot \frac{\varepsilon^4}{(-V_{-1})^2(-V_{-2})}$$

In general, we get "a tree of loops".

Exercise: how many different pictures are there at ε^n?

How many total terms?

- Arnol'd V., *Remarks on perturbation theory for problems of Mathieu type* (Russian), Uspekhi Mat. Nauk 38 (1983), no. 4 (232), 189 – 203.



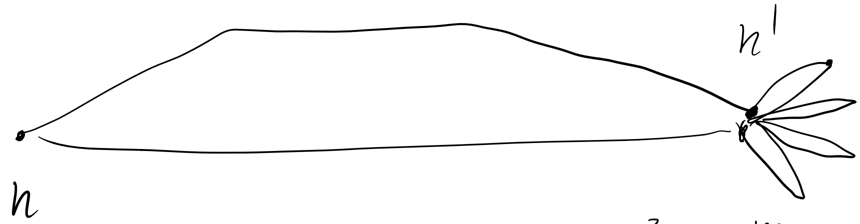
Convergence $\quad V_n \to 0$

Apply these procedures for each V_n. The formal series makes sense if $V_n \neq V_m$ for $n \neq m$.

V_n isolated $\Longrightarrow$ always converges.

V_n not isolated $\Longrightarrow$ individual terms diverge.

$$|V_{n'} - V_n| < \varepsilon^{100}$$



$$\left(\frac{\varepsilon^2}{|V_n - V_n'|}\right)^m \longrightarrow \infty$$

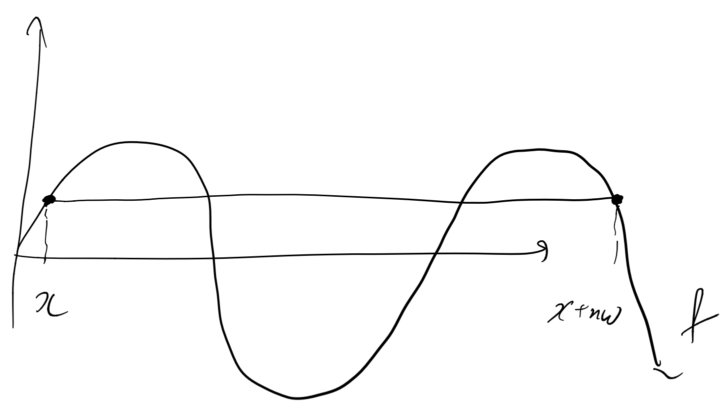
Let $V_n = f(x+n\omega)$ $\qquad f(x) = f(x+1)$

$V_0 = f(x)$

$$V_n^{-1} = \frac{1}{f(x) - f(x+n\omega)}$$

Small denominator: $\|n\omega\| \lesssim \varepsilon$

Resonance:



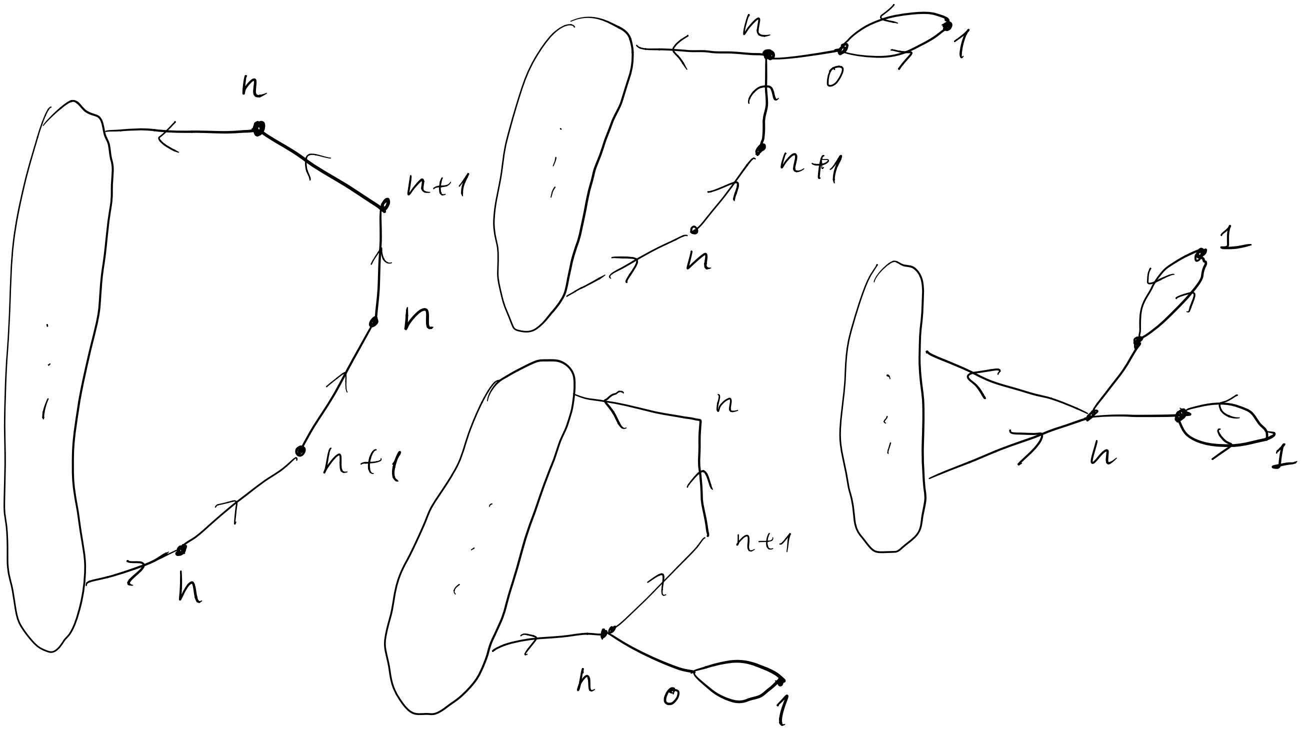
Maryland-type:

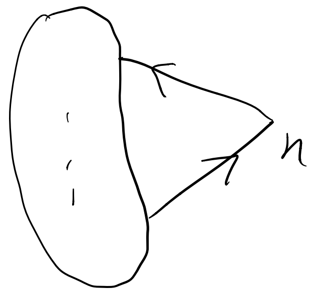
no resonances, only small denominators!

Problem: short segments that return to the same small denominator multiple times.

Let $|f(x) - f(x+n\omega)| \leq \varepsilon^{100}$.

Idea: cancel with matching attached configurations.



Let $h(x) =$ 

$$h(x)\left\{\frac{1}{(V_0-V_n)(V_0-V_{n+1})(V_0-V_n)(V_0-V_{n+1})}-2\frac{1}{(V_0-V_n)^2(V_0-V_{n+1})(V_0-V_1)}+\right.$$

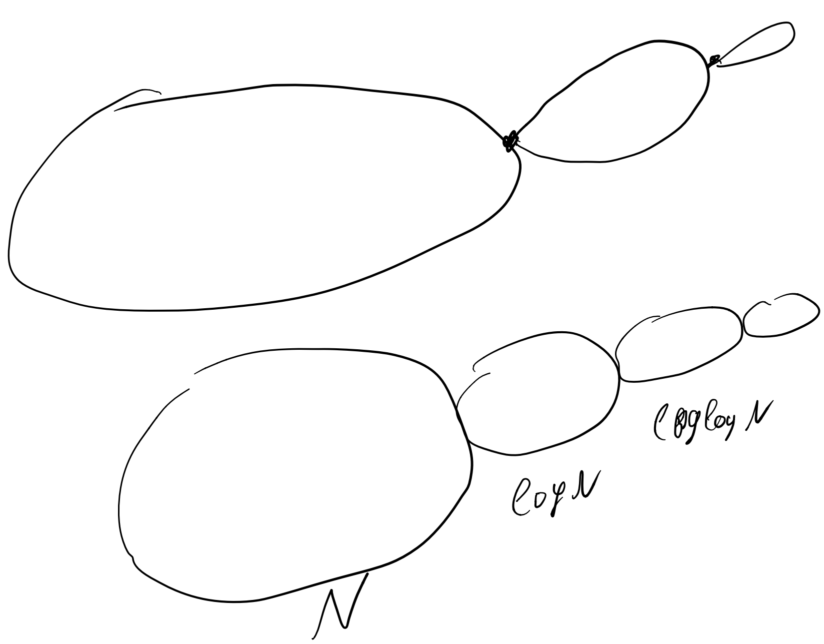
$$\left.+\frac{1}{(V_0-V_n)^2(V_0-V_1)^2}\right\} =$$

If $\|n\omega\|$ is small, then $V_0 - V_1$ is close to $V_0 - V_{n+1}$, and neither is small!

$$= h(x) \cdot \frac{1}{(V_0 - V_n)^2} \cdot \left(\frac{1}{V_0 - V_1} - \frac{1}{V_0 - V_{n+1}} \right)^2 =$$

$$= h(x) \cdot \left(\frac{f(x+\omega) - f(x+(n+1)\omega)}{f(x) - f(x+n\omega)} \right)^2 \cdot \frac{1}{(f(x) - f(x+\omega))(f(x) - f(x+(n+1)\omega))}$$

no more ε^{100}



$\varepsilon << 1$

Note: relation to KAM

- J. Bellissard, R. Lima, and E. Scoppola, *Localization in ν-dimensional incommensurate structures*, Comm. Math. Phys. 88 (1983), no. 4, 465 – 477.

$$H = V + \varepsilon \Delta$$

$$H_1 = e^{i\varepsilon A_1} H e^{-i\varepsilon A_1}$$

$$A_1 = A_1^*$$

$$\left(1 + i\varepsilon A_1 + O(\varepsilon^2)\right)(V + \varepsilon\Delta)\left(1 - i\varepsilon A_1 + O(\varepsilon^2)\right) =$$

$$= V + \varepsilon\left(\Delta + i[A_1, V]\right) + O(\varepsilon^2)$$

Choose A_1 s.t. $i[A_1, V] = -\Delta$

$(A_1)_{ii} = 0$ (in the basis of eigenvectors of V)

$$(A_1)_{ij} = \frac{(-\Delta)_{ij}}{i(V_i - V_j)} = \frac{1}{i(V_i - V_{i+1})}\delta_{i+1,j} + \frac{1}{i(V_i - V_{i-1})}\delta_{i,j+1}$$

Result: $H_1 = V_1(\varepsilon) + \varepsilon^2 \Delta_1(\varepsilon)$

Repeat. $\varepsilon \to \varepsilon^2 \to \varepsilon^4 \to \varepsilon^8 \to \dots$ ("superconvergence")

- Eliasson H., *Absolutely convergent series expansions for quasi periodic motions*, Math. Phys. Electron. J. 2 (1996), Paper 4, 33 pp.