Quasiperiodic operators with monotone potentials

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https://users.math.msu.edu/users/ikachkov/minicourse.html

Plan of Lecture 1:

1) Schrodinger equation and spectral theorem
2) Types of spectra and quantum dynamics
3) Anderson localization
4) Ergodic operators, spectral invariance
5) Quasiperiodic operators, examples
6) Lyapunov exponent, density of states, Kotani theory, Thouless formula.

7) Schnol's Theorem and sufficient conditions for localization

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① Motivation: the Schrödinger equation

Let $\mathcal{H}$ be a Hilbert space ($\ell^2(\mathbb{Z})$; later $\ell^2(\mathbb{Z}^d)$)

$H = H^*$ a self-adjoint operator ("$\langle H\varphi, \psi\rangle = \langle \varphi, H\psi\rangle$").

$$\begin{cases} \psi(0) = \psi_0 \in \mathcal{H} \\ -i \dfrac{\partial \psi}{\partial t} = H\psi \end{cases}$$

Solution:

$$\psi(t) = e^{iHt}\psi_0$$

② What is e^{iHt}?

If $\dim \mathcal{H} < +\infty$, then

$$H = \sum_j \lambda_j P_j = \int_{\mathbb{R}} \lambda \, d\mathbb{E}(\lambda)$$

where $\mathbb{E} = \sum P_j \delta_{\lambda_j}$ is (a simple case of) a spectral measure.

$$e^{itH} = \sum_j e^{it\lambda_j} P_j = \int e^{it\lambda} d\mathbb{E}(\lambda)$$

③ The spectral theorem.

Let $H = H^*$. Then

$$H = \int_{\mathbb{R}} \lambda \, d\mathbb{E}(\lambda)$$

Where

$\mathbb{E}$ is a Borel measure on $\mathbb{R}$, whose values are orthogonal projections.

$\mathbb{E}(\delta_1 \cap \delta_2) = \mathbb{E}(\delta_1)\mathbb{E}(\delta_2)$; δ_j Borel sets.

$\mathbb{E}(\mathbb{R}) = I$.

Let $\varphi, \psi \in \mathcal{H}$. Then

$$\left.\begin{aligned} \mu_{\varphi,\psi}(\delta) &= \langle \mathbb{E}(\sigma)\varphi, \psi \rangle \\ \mu_{\varphi}(\delta) &= \langle \mathbb{E}(\delta)\varphi, \varphi \rangle \end{aligned}\right\} \text{ scalar measures.}$$

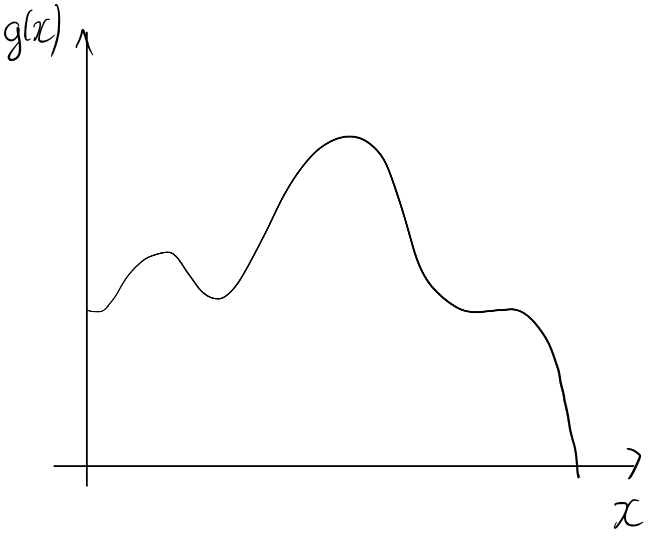
Alternatively, we can say:

$$\int f(E)\, d\mu_{\varphi,\psi}(E) = \langle f(H)\varphi, \psi \rangle$$

Example: multiplication operator on $L^2(\mathbb{R})$:

$$(M_g f)(x) = g(x) f(x)$$

$$\mathbb{E}_{M_g}(\delta) = \mathbb{1}_{\{x : g(x) \in \delta\}}$$



$$\sigma(M_g) = \text{ess Ran}(g)$$

④ Spectral types

Let $H = H^*$ on $\mathcal{H}$. Then

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \,,$$ where

$\mathbb{E}\big|_{\mathcal{H}_j}$ has the corresponding type:

purely point, absolutely continuous, singular continuous

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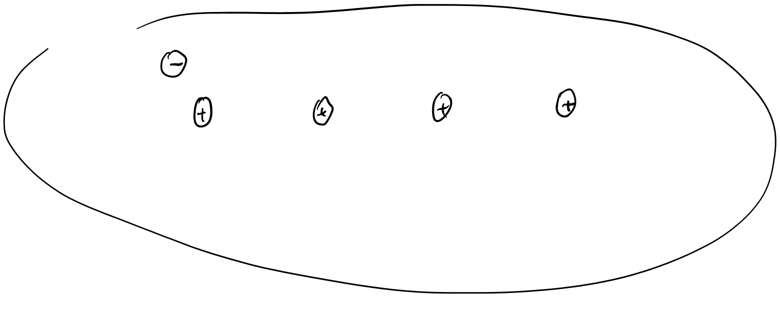
⑤ Schrödinger operators: tight binding $\mathcal{H} = \ell^2(\mathbb{Z})$

$$(H\psi)_n = \underbrace{\psi_{n+1} + \psi_{n-1}}_{\text{hopping terms}} + \underbrace{V_n}_{\text{potential}} \psi_n = (\underset{\substack{\uparrow \\ \text{discrete Laplace operator}}}{\Delta}\psi)_n + (V\psi)_n$$

Motivation: electron stays close to an ion, but can "hop" between ions



⑥ Quantum dynamics: how "spread out" is $e^{iHt}\psi_0$ if ψ_0 is localized (say, $\psi_0 = e_0$)

For example: let $q(t) = \sqrt{\sum_{n\in\mathbb{Z}} |n|^2 |\psi_n(t)|^2}$

expectation value of the coordinate (2nd moment)

ac spectrum: $q(t) \sim t$

pp spectrum: $q(t) \sim \text{const}$

sc spectrum: $q(t) \sim t^{\alpha}$, $0<\alpha<1$ (sometimes).

Related to growth properties of the Fourier transform of the spectral measure:

$$\langle \psi(t), \psi(0)\rangle = \int e^{it\lambda}\langle dE(\lambda)\underset{\psi(0)}{\psi}, \psi\rangle = |\hat{\mu}_{\psi}(t)|^2$$

Exercises: ① a) $(H\psi)_n = \psi_{n+1} + \psi_{n-1}$.

Let $\delta \subset \mathbb{R}$ be an interval. Calculate $\mathbb{E}_H(\Delta)$ (as an operator on $\ell^2(\mathbb{Z})$.

b) On $L^2(\mathbb{R}^n)$, consider $H\psi = -\Delta\psi = \sum_j \frac{\partial^2\psi}{\partial x_j^2}$.

Same question.

② Let $\psi \in \ell^2(\mathbb{Z})$ be compactly supported

Prove $q(t) \geq C(\psi)|t|$ (for $H = \Delta$)

⑦ Anderson localization: let $H = \Delta + V$.
We say that H satisfies AL is $\sigma(H)$ is purely point:
$H\psi_j = E_j \psi_j$, and $|\psi_j(n)| \le c_j e^{-\gamma_j |n-n_j|}$
(sometimes $\gamma_j \ge \gamma > 0$).

Discovered by P.W. Anderson in 1958, for random potentials: $\{V_n\}_{n\in\mathbb{Z}}$ are i.i.d.r.v., non-constant.

$d=1$: known in, essentially, all cases.

$d\ge 2$: open question for small disorder and/or Bernoulli-type distributions.

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⑧ AL can be observed not only for random, but for other <u>ergodic operators</u>.

Let (Ω, μ) be a measure space, $\mu(\Omega) = 1$, and $T: \Omega \to \Omega$ be a measure preserving bijection:
$\mu(T(A)) = \mu(T^{-1}(A)) = \mu(A)$.

Assume that T is <u>ergodic</u>:

$$T(A) = A \Longrightarrow \mu(A) \in \{0, 1\}.$$

(no non-trivial invariant subsets)

Let $f: \Omega \to \mathbb{R}$ be a measurable function.

Exercise: ③ Let $T: \Omega \to \Omega$ ergodic

$f: \Omega \to \mathbb{R}$ measurable,

T-invariant: $f(T\omega) = f(\omega)$.

Show that $f(\omega) = \text{const}$ a. e.

$T: \Omega \to \Omega$ ergodic

Let $f: \Omega \to \mathbb{R}$ be a measurable function.

$V_n(\omega) = f(T^n\omega)$ is an <u>ergodic potential</u>

$(H(\omega)\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + f(T^n\omega)\Psi_n$ is an ergodic Schrödinger operator.

Examples: 1) Random operators:

$$\Omega = (\Omega_1)^{\mathbb{Z}}, \quad (T\omega)_m = \omega_{m+1}, \quad \omega = \{\omega_n\}_{n\in\mathbb{Z}}.$$

single site distribution

2) Quasiperiodic operators (our focus):

$$\Omega = \mathbb{T}^1 = [0,1). \quad Tx = (x+\alpha)(\mathrm{mod}\,\mathbb{Z}), \quad \alpha \in \mathbb{R}\setminus\mathbb{Q}.$$

Exercise: prove T is ergodic.

$$(H(x)\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + f(x+n\alpha)\Psi_n.$$

Invariance of the spectra:

$H(x+\alpha) = U^{-1}H(x)U$, $(U\Psi)_n = \Psi_{n+1}$.

Th (Kunz-Souillard ?)) : there are $\Sigma, \Sigma_{ac}, \Sigma_{pp}, \Sigma_{sc} \subset \mathbb{R}$ s.t. $\sigma_\bullet(H(x)) = \Sigma_\bullet$ for a.e. x.

Idea of the proof: Fix $p, q \in \mathbb{Q}$.

Let $$g^\bullet(x) = \begin{cases} 1, & \mathbb{E}^\bullet((p,q)) \neq 0 \\ 0, & \mathbb{E}^\bullet((p,q)) = 0. \end{cases}$$

$g^\bullet$ is an invariant function on $\mathbb{T}^1$ (immediately)

$g^\bullet$ is measurable: not trivial, use quantum dynamics

Exercise: ④ a) Let $f:\mathbb{T}^1\to\mathbb{R}$ be continuous. Show that $\sigma(H(x)) = \text{const}$ for all (not just a.e.) $x\in\mathbb{T}^1$

b) Same for $f(x) = \mathbb{1}_{[0,a)}(x)$, $0<a<1$.

Th For any fixed $E\in\mathbb{R}$, (Pastur?)

$$\left|\{x\in\mathbb{T}^1 : E \text{ is an eigenvalue of } H(x)\}\right| = 0.$$

Therefore, while $\sigma_{pp} \overset{a.s.}{=} \text{const}$, the eigenvalues "move"

Meta-result: there exists a function $E:\mathbb{T}^1\to\mathbb{R}$ s.t. $\{E(x+n\alpha) : n\in\mathbb{Z}\}$ are the eigenvalues of $H(x)$ for a.e. x.

E is called the Sinai function.

Diophantine properties

In QP operators, Diophantine properties of α become important.

Let $DC_d(c,\tau) = \{\alpha\in\mathbb{T}^d : \|n\cdot\alpha\|\geq c|n|^{-\tau}\}$. $\|a\| = \text{dist}(a,\mathbb{Z})$ $n\neq 0$.

Exercise: Fix $\tau > d+1$. Then $\bigcup_{c>0} DC(c,\tau)$ has full Lebesgue measure.

continued fraction expansion

Let $d=1$, $\frac{p_n}{q_n}\longrightarrow\alpha$. $\beta(\alpha) := \limsup_{n\to\infty}\frac{\ln q_{n+1}}{q_n}$

$DC \Rightarrow \beta(\alpha)=0$

the Almost Mathieu Operator

$(H(x)\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + 2\lambda \cos 2\pi(x+n\alpha)$, $n \in \mathbb{Z}$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

$|\lambda| < 1$ purely ac spectrum, all x

$|\lambda| = 1$ purely sc spectrum, all x (recent)

$1 < |\lambda| < e^{\beta(\alpha)}$ purely s.c. for a.e. x

$|\lambda| > e^{\beta(\alpha)}$ pp spectrum for a.e. (but not every) x

There is an additional transition in x.

Most recent results (see references): Avila-You-Zhou, Jitomirskaya-Liu

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the Maryland model

$$(H(x)\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + \lambda \tan \pi(x+n\alpha)\,\Psi_n$$

Let $\alpha \in DC$, $x \in \mathbb{R} \setminus (\mathbb{Z} + \alpha\mathbb{Z})$. Then $\sigma(H(x))$ is purely point, for all $\lambda > 0$.

The proof also works on $\mathbb{Z}^d$, but relies heavily on the tan structure.

In $d=1$, the spectral type is described completely for <u>all</u> λ, x, α (Jitomirskaya-Liu)

Transfer matrices

$H(x) = \Delta + f(x+n\omega)$

Let $M(x) = \begin{pmatrix} E-f(x) & -1 \\ 1 & 0 \end{pmatrix}$ $\leftarrow$ transfer matrix

$$M_n(x) = \prod_{k=n}^{1} \begin{pmatrix} E - v(x+(k-1)\omega) & -1 \\ 1 & 0 \end{pmatrix} \leftarrow \text{n-step transfer matrix}$$

$$H(x)\psi = E\psi \iff \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} = M_n(x) \begin{pmatrix} \psi_0 \\ \psi_{-1} \end{pmatrix}, \forall n \in \mathbb{Z}$$

Let
$$H_n(x) = \begin{pmatrix} f(x) & 1 & & & \\ 1 & f(x+\omega) & 1 & & 0 \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & f(x+(n-1)\omega) \end{pmatrix}$$

$$P_n(x,E) := \det(H_n(x) - E)$$

Exercise: $$M_n(x,E) = \begin{pmatrix} P_n(x,E) & -P_{n-1}(x+\alpha,E) \\ P_{n-1}(x,E) & -P_{n-2}(x+\alpha,E) \end{pmatrix}$$

Lyapunov exponent:

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln\|M_n(x,E)\| dx = \inf_n \frac{1}{n} \int_0^1 \ln\|M_n(x,E)\| dx$$

Th (Kingman) Let $F_N : T^1 \longrightarrow \mathbb{R}$ be a subadditive process:

$$F_{N+M}(x) \leq F_N(x) + F_M(x + n\alpha)$$

$$\int |F_N(x)|\, dx < +\infty, \quad \inf_N \frac{\int |F_N(x)|\, dx}{N} =: \Gamma(F) > -\infty$$

Then $\frac{1}{N} F_N$ converges a.s. to $\Gamma(F)$.

Apply to $F_N = \ln \| M_N(x, E) \|$.

Th (Ishii-Pastur-Kotani):

$$\sigma_{ac}(H) = \overline{\{E : L(E) = 0\}}^{ess}.$$

Integrated density of states

Let $N_n(x, E) = \# \sigma(H_n(x)) \cap (-\infty, E]$ (counting function).

$$N(E) := \lim_{n \to \infty} \frac{1}{N} \int_0^1 N_n(x, E)\, dx$$

The limit exists for very general ergodic operators.

$E \mapsto N(E)$ is a continuous non-decreasing function, and is the distribution function of the density of states measure:

$$\int_{\mathbb{R}} f(E)\, dN(E) = \int_{\mathbb{R}} f(E) \left(\int_0^1 \langle \mathbb{E}_{H(x)} e_0, e_0 \rangle dx \right)$$

DoSM is the expectation value of the spectral measures.

Theorem (Thouless formula)

$$L(E) = \int \ln|E-E'|\, dN(E')$$

Lyapunov exponent

DoSM

Heuristics: $L(E) \text{"}\approx\text{"} \frac{1}{n} \int_0^1 \ln|P_n(x,E)|\, dx =$

$$= \frac{1}{n} \int_0^1 \sum_{j=0}^{n-1} \ln|E - E_j(x)|\, dx \eqsim \int_0^1 \ln|E-E'|\, N_n(dE')$$

counting measure

This can be made rigorous by considering $E \in \mathbb{C}^+$ and showing that L is subharmonic on $\mathbb{C}^+$.

Let $(H\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + V_n \Psi_n$ (general operator)

We say that Ψ is a generalized eigenfunction

if
- $|\Psi(n)| \leq C_1(1+|n|)^{C_2}$ (polynomially bounded)
- $H\Psi = E\Psi$ for some $E \in \mathbb{R}$
  ↖ generalized eigenvalue.

In general, $\Psi \notin \ell^2(\mathbb{Z})$.

Theorem (Шноль)) The spectral measure of H is supported on the set of generalized eigenvalues of H.

Many modern proofs of AL use Green's function method.

Let $H_N = H|_{[0,N-1]}$, $H\Psi = E\Psi$.

Then
$$H_N\Psi = E\Psi - \Psi_{-1}e_0 - \Psi_N e_{N-1}$$

$$\Psi_m = -(H_N - E)^{-1}_{0m}\Psi_{-1} - (H_N - E)^{-1}_{m,N-1}\Psi_N$$

Then, try to prove that $(H_N - E)^{-1}$ has exponential decay away from the diagonal.