Many versions of the uncertainty principle, Lectures December 2020

Lecture 1. Problems, Notes and Further reading

We discussed the Heisenberg uncertainty principle, a related theorem of Balian and Low and a construction of Bourgain of a well localized basis. Then we proved the Hardy and Beurling uncertainty principles.

Problem 1. Let $h_n = c_n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}$ be the sequence of Hermite functions. Each function h_n can be written as $h_n = P_n(x)e^{-x^2/2}$, where P_n is a polynomial of degree n.
(a) Show that $\widehat{h_n} = (-i)^n h_n$ and $Hh_n = (2n+1)h_n$, where $Hf = -f'' + x^2 f$ is the Hermite operator.
(b) Let $\Delta^2 f = \int_{\mathbb{R}} |x|^2|f(x)|^2dx$. Suppose that $\{f_j\}_{j=1}^k$ is an orthonormal sequence in $L^2(\mathbb{R})$. Show that

$$\sum_{j=1}^k(\Delta^2 f_j + \Delta^2 \widehat{f_j}) \geq k^2.$$

Problem 2. suppose that $u \in C^1([0,T], W^{2,2}(\mathbb{R}^d))$ is a solution of the free Schrödinger equation $\partial_t u(t,x) = i\Delta_x u(t,x)$ that satisfies the following decay conditions

$$|u(0,x)| \leq Ce^{-\alpha|x|^2} \text{ and } |u(T,x)| \leq Ce^{-\beta|x|^2},$$

with $\alpha, \beta > 0$. We consider the case $\alpha\beta = (16T^2)^{-1}$. Let

$$F(z,x) = \frac{1}{\sqrt{4\pi z}}\int_{\mathbb{R}} e^{-|x-y|^2/(4z)}u_0(y)dy$$

(a) Let z_0 be the common point of the circles $\omega_0 = \{z \in \mathbb{C} : |z + 1/(8\alpha)| = 1/8\alpha\}$ and $\omega_1 = \{z \in \mathbb{C} : |z - iT + 1/8\beta| = 1/(8\beta)\}$. Show that the function $F^2(z,0)$ has a simple pole at z_0 and tends to zero at infinity. (It is easy to show that the pole is of order at most two, a more careful consideration shows that the pole is of order one.)
(b) Similarly to (a) one can show that $\partial_x F^2(z,0)$ is a holomorphic function with a pole at z_0 of order not larger than 2. Finally, compare the functions $F(z,0)$ and $F_x(z,0)$ to the corresponding function obtained for $\tilde{u}_0 = c_0h_0 + c_1h_1$ and finish the proof of the part (ii).

Problem 3. The Beurling uncertainty principle says that if $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||f(\xi)|e^{|x\xi|}\,dx\,d\xi < \infty$$

then $f = 0$ in $L^1(\mathbb{R})$.
(a) Show that it is enough to prove the result for the case when f is even or odd and real valued.
(b) Consider the function

$$F(s) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)f(\xi)e^{isx\xi}.$$

Show that it can be extended to a holomorphic function on the whole complex plane.
(c) Conclude the proof of Beurling's theorem.

Notes. The main results of the lecture are classical. The Balian-Low theorem was proved independently by Balian and Low in the 1980s, we refer to a modern exposition [10] and the references therein. For the details of Bourgain's construction see [3], a further discussion and some open problems in higher dimensions can be found in [6]. The original work of Hardy [7], as mentioned during the lecture, contains two proofs of the theorem. The second one is based on the analysis of the function $s(\lambda) = \int_0^\infty f(x)e^{-\lambda x^2/2}dx$, where f is assumed to be either odd or even and to satisfy the additional symmetry condition, $\hat{f} = cf$. The dynamical proof is given in [4]. The two known proofs of the Beurling uncertainty principle are in [9] and [8], see also [2] for higher dimensional versions and [5] for a further generalization of Hedenmalm's result.

References

[1] A. Beurling, The Collected Works of Arne Beurling, vol.2, Harmonic Analysis, Birkhäuser, 1989.

[2] A. Bonami, B. Demange, P. Jaming, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, Rev. mat. Iberoam., 19 (2003), 23–55.

[3] J. Bourgain, A remark of the uncertainty principle for Hilbertian basis, J. Funct. Anal., 79 (1988), 136–143.

[4] A. Fernández-Bertolin, E. Malinnikova, Dynamical versions of Hardy's uncertainty principle: a survey, preprint, 2020.

[5] X. Gao, On Beurling's uncertainty principle, Bull. London Math. Soc. 48 (2016), no. 2, 341–348.

[6] K. Gröchenig, Malinnikova, Phase space localization of Riesz bases for $L^2(\mathbb{R}^d)$, Rev. Mat. Iberoam. 29 (2013), 115–134.

[7] G. H. Hardy, A theorem concerning Fourier transforms, J. London Math. Soc., 8 (1933), 227–231.

[8] H. Hedenmalm, Heisenberg's uncertainty principle in the sense of Beurling, J. Anal. Math. 118 (2012), 691–702.

[9] L. Hörmander, A uniqueness theorem of Beurling for Fourier transform pairs, Ark. Mat. 29 (1991), 237–240.

[10] S. Nitzan, J.-F. Olsen, A quantitative Balian-Low theorem, J. Fourier Anal. Appl., 19 (2013), 1078–1092.