Many versions of the uncertainty principle, Lectures December 2020

Lecture 2. Problems, Notes and Further reading

We discussed the Benedicks uncertainty principle and its quantitative versions, including an inequality of Nazarov. Then we proved Kovrijkine's version of the Logvinenko–Sereda theorem, and formulated recent results of Bourgain and Dyatlov.

Problem 1. We outline another proof of the Benedicks theorem in this problem.
(a) Assume that $A, B \subset \mathbb{R}$ are sets of finite measure. Let $P_A Q_B f = \chi_A(\mathcal{F}^{-1}(\chi_B \mathcal{F} f))$. Consider the space $X = \{\phi : P_A Q_B \phi = \phi\}$. Show that the dimension of X is bounded by $|A||B|$.
(b) Assume that A and B have finite measures and there is a function f such that f is supported in A and $\widehat{f}$ is supported in B. Construct a sequence of sets $A = A_1 \subset A_2 \subset A_3 \subset ...$ and a sequence $f_k(x) = f(x - x_k)$ such that f_k is supported in A_k, $\widehat{f_k}$ is supported in B and $|A_k| < |A_{k-1}| + 2^{-k}$. Use part (a) to get a contradiction.

Problem 2. Let E, F be sets of finite measure. Show that for any functions $f, g \in L^2(\mathbb{R})$ there exists a function $h \in L^2(\mathbb{R})$ such that $h = f$ on E and $\widehat{h} = g$ on F.

Problem 3. Prove the following simplified version of the Logvinenko-Sereda theorem. Let $supp(\widehat{f}) \subset (-1, 1)$ and assume that $S \subset \mathbb{R}$ is such that $S \cap (n, n+1) < \alpha < 1$ for each $n \in \mathbb{Z}$. Show that $\|f\|_{L^2(S)} \le \delta(\alpha)\|f\|_{L^2(\mathbb{R})}$ for any $f \in L^2(\mathbb{R})$, where $\delta(\alpha) \to 0$ when $\alpha \to 0$.
(Hint: Let ϕ be a functions with $\widehat{\phi} = 1$ on $(-1, 1)$ and smooth. Consider $K_S(x, y) = \chi_S(x)\phi(x - y)$ and show that K_S is a kernel of an operator with small norm when α is small.)

Notes. The theorem of Benedicks dates back to his preprint of 1974. A different proof, outlined in Problem 1, was given by Amrein and Berthier in 1977, see [1]. Benedicks argument was published in [2]. A detailed discussion of related results is given in [7, Chapter 3]. I strongly recommend everyone to read a beautiful article by Fedya Nazarov [11]. A higher dimensional version of the Nazarov estimate is given in [5]. The proof of the quantitative version of the Amrein– Berthier theorem can be found in chapter 2 of this work, which also contains many related results, including results on one-sided decay. In our discussion of the Logvinenko–Sereda theorem, which originally appeared in [9], we followed the proof of Oleg Kovrijkine, see [8]. One can consult [8] or [10, Chapter 10] for further details. The fractal uncertainty principle that we discussed is due to Bourgain and Dyatlov, see [3], explicit bounds can be found in [6]. We also refer to [4] for detailed discussion and applications.

References

[1] W. O. Amrein and A.M. Berthier, On support properties of L^p-functions and their Fourier transforms, J. Funct. Anal., 24 (1977), 258–267.

[2] M. Benedicks, On Fourier transforms of functions supported on sets of finite Lebesgue measure, J. Math. Anal. and Appl., 106 (1985), 180–183.

[3] J. Bourgain, S. Dyatlov, Spectral gaps without the pressure condition, Ann. Math. 187 (2018), 1–43.

[4] S. Dyatlov, An introduction to fractal uncertainty principle, J. Math. Phys., 60 (2019), 081505.

[5] P. Jaming, Nazarov's uncertainty principle in higher dimensions, J. Approx. Theory, 149 (2007), 30–41.

[6] L. Jin, R. Zhang, Fractal uncertainty principle with explicit exponent, Math. Ann., 376 (2020), 1031–1057.

[7] V. Havin, B. Jöricke, The uncertainty principle in harmonic analysis, Springer-Verlag, 1994, chapter 3.

[8] O. Kovrijkine, Some results related to the Logvinenko-Sereda theorem, Proc. AMS, 129 (2001), 3037–3047.

[9] V. N. Logvinenko, Ju. F. Sereda, Equivalent norms in the spaces of entire functions of exponential type, Teor. Funkcii Funk. Anal. i Pril., 20 (1974), 102–111.

[10] C. Muscalu, W. Schlag, Classical and Multilinear Analysis, vol. 1, Cambribde Univ. Press, Chapter 10.

[11] F. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type, St. Petersburg Math. J., 5 (1994), 663–717. (Algebra i Analiz, 5 (1993), no. 4, 3–66.)