Many versions of the uncertainty principle, Lectures December 2020

Lecture 3. Problems, Notes and Further reading

We discussed some generalizations of Heisenberg's uncertainty and the the results related to the celebrated $2TW$ theorem of Landau, Slepian, and Pollak

Problem 1. Prove the following statement of Strichatz: Let $\{a_j\}_{-\infty}^{\infty}$ be a sequence of points with $a_{j+1} - a_j \leq b$, $\lim_{j\to\pm\infty} a_j = \pm\infty$, and let $f \in W^{1,2}(\mathbb{R})$. If $\sum_j |f(a_j)|^2 \leq (1-\epsilon)^2 b^{-1/2} \|f\|_{L^2}^2$ then $\|f'\|_{L^2}^2 \geq c\epsilon^2 b^{-2} \|f\|_{L^2}^2$. Compare to the statement of (the weak version of) the Logvinenko-Sereda theorem.

Problem 2. Let $\{\phi_n\}$ be an orthonormal basis for $L^2(\mathbb{R})$ of eigenfunctions of the self-adjoint operator $Q_B P_A Q_B$.
(a) (Double orthogonality property) Show that $\int_A \phi_k \phi_j = \lambda_J \delta_{jk}$.
(b) Let $A = [-1, 1]$ and $B = [-c, c]$. Show that the restriction operator commutes with the differential operator $(1-x^2)f'' - 2xf - cx^2 f$ and that the latter has simple eigenvalues.

Problem 3. The Poisson summation formula follows from considering the periodic function $F(x) = \sum f(x+n)$ and its Fourier transform. Prove the Shannon interpolation formula:

$$f(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{\pi k}{W}\right) \frac{\sin(Wt - \pi k)}{Wt - \pi k},$$

for any f such that $supp(\widehat{f}) \subset (-W, W)$.

Notes. There are many generalizations of the classical inequalities to L^p settings. We mentioned the one by Cowling and Price [2], other relevant results are Bourgain's construction [1] and Balian-Low theorem [4] in L^p, L^2 setting. The Strichartz lemma and much more general uncertainty results are contained in [15]. A version for the Besov spaces and further sampling formulas can be found in [6].

The prolate spheroid wave functions in time-frequency localizations appeared in the series of works of H. Landau, H. Pollak, and D. Slepian, in the early 1960s, [8, 13, 14, 9]. An interesting approach to the $2TW$ theorem is given in [5].

The Poisson summation formula and Shannon interpolation formula are classical and can be found in many textbooks. The last result we mentioned on the lecture is the remarkable interpolation formula by Radchenko and Viazovska, see [12]. See also [7] where a connection to the $2TW$ theorem is discussed.

A nice surveys of uncertainty principles from the end of the last century is [3].

References

[1] J. Benedetto and A. Powell, A (p, q) version of Bourgain's theorem, *Trans. Amer. Math. Soc.*, 358 (2006), no. 6, 2489–2505.

[2] M. G. Cowling & J. F. Price, *Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality*, SIAM J. Math. Anal. 15 (1984), 151–165.

[3] G. B. Folland and A. Sitaram, The Uncertainty Principle: A Mathematical survey, *J. Fourier Anal. and Appl.*, 3 (1997), no. 3, 208–238.

[4] S. Z. Gautam, A critical-exponent Balian-Low theorem, *Math. Res. Letters*, 15 (2008), no. 3, 471-483.

[5] A. Israel, The eigenvalue distribution of time-frequency localization operators, arXiv:1502:04404

[6] P. Jaming and E. Malinnikova, An uncertainty principle and sampling inequalities in Besov spaces, J. Fourier Anal. Appl., 22 (2016), 768–786.

[7] A. Kulikov, Free interpolation and time-frequency localization, arXiv:2005.12836.

[8] H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals, *Bell System Tech. J.*, 41 (1962) 1295–1336.

[9] H. J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.*, 117 (1967), 37–52.

[10] H. J. Landau, H. Widom, Eigenvalue distribution of time and frequency limiting, J. Math. Anal. Appl., 77 (1980), 469–481.

[11] A. Osipov, Certain upper bounds on the eigenvalues associated with prolate spheroidal wave functions, Applied and Comp. Harm. Anal., 35 92013), 309–340.

[12] D. Radchenko, M. Viazovska, Fourier interpolation on the real line, Publ. Math. Inst. Hautes Études Sci. 129, (2019), 51–81.

[13] D. Slepian and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty, I, *Bell System Tech. J.*, 40 (1961), 43–64.

[14] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertanity. IV. Extensions to many dimensions; generalized prolate spheroidal functions. *Bell System Tech. J.*, 43 (1964) 3009–3057.

[15] R. S. Strichartz, *Uncertainty principles in harmonic analysis*, J. Funct. Anal., 84 (1989), 97–114.