Lecture 1. Expanders

Let Γ be a connected graph on N vertices. Consider the random walk $(Z_n)_n$ on Γ determined by the initial distribution π and by the matrix $P = (p(x,y))_{x,y\in V(\Gamma)}$ of transition probabilities, with $p(x,y) = a(x,y)/deg(x)$ where $A = (a(x,y))_{x,y\in V(\Gamma)}$ is the adjacency matrix of the graph and $deg(x)$ denotes the degree of the vertex x. Let μ be the stationary distribution of the random walk.
1. Show that

$$\|P\pi - \mu\|_2 \leq \alpha_2,$$

where α_2 is the first nontrivial eigenvalue of the matrix P.

2. Let $B \subset V(\Gamma)$, $t \in \mathbb{N}$. Show that

$$\mathbb{P}\{Z_0 \in B, ..., Z_t \in B\} \leq (\frac{|B|}{N} + \alpha_2)^t.$$

3. Define, similarly to the laplacian $\Delta = d^*d$ acting on $C^0(\Gamma)$, the edge-laplacian $\Delta_1 = dd^*$ acting on the space $C^1(\Gamma)$ of 1-forms, or functions on the edges of the graph. Prove the Hodge decomposition :

$$C^1(\Gamma) = Ker(\Delta_1) \oplus Im(d).$$

The subspace $Ker(\Delta_1)$ is called the cycle subspace and the subspace $Im(d)$ is called the cut subspace of Γ. Describe the corresponding bases of these subspaces. Show the following classical result called Matrix-Tree Theorem :

The product of non-zero eigenvalues of Δ is equal to the product of non-zero eigenvalues of Δ_1 is equal to $|V(\Gamma)|\kappa(\Gamma)$, where $\kappa(\Gamma)$ is the complexity of the graph, i.e. the number of spanning trees in Γ.

Let Γ be a finite connected graph. Prove that

$$\log \kappa(\Gamma) = -\log(2|E(\Gamma)|) + \sum_{x\in V(\Gamma)} \log deg(x) - \sum_{k\geq 1} \frac{1}{k}(\sum_{x\in V(\Gamma)} p_k(x,x) - 1),$$

where $p_k(x,x)$ denotes the probability that the random walk starting at x comes back to x after k steps.

4. Here is a historical problem solved by Lagrange in 1759. He was interested in finding the wave equation that would describe the sound propagation inside organ pipes. For modelling this situation he was using the model with a very big number n of coupled harmonic oscillators. His solution gives in particular the chronologically first known example of spectral computations for square matrices of arbitrary size.

For each $j = 1, ..., n$, the j-th mass is at disctance $x_j(t)$ from its equilibrium. It is acted upon by forces proportional to $-(x_j(t) - x_{j-1}(t))$ from the left and to $x_{j+1}(t) - x_j(t)$ from the right. Newton equations then lead to the following system of linear differential equations with constant coefficients :

$$\ddot{x}_j(t) = K(x_{j-1}(t) - 2x_j(t) + x_{j+1}(t)), \quad j = 1, ..., n$$

where K is a positive constant. Solution for $n = 1$ suggests to look for solutions of the form

$$y_j(t) = x_j(t)\cos(\omega t + t_0) \ .$$

Solve the system.