Lecture 2. Amenability

1. Compute the isoperimetric constant and the spectral radius of the infinite k-regular tree.

2. For an infinite connected regular graph, prove $i(\Gamma) = 0$ iff $\rho(\Gamma) = 1$ via appropriate versions of Cheeger-Buser inequalities.

3. Recall that the Lamplighter group $L_{2,1}$ is defined as follows : the elements of the group are couples $l = (\alpha, h)$, where $\alpha : \mathbb{Z} \to \mathbb{Z}_2$ is a function with finite support $|supp(\alpha)| < \infty$ and $h \in \mathbb{Z}$ is an integer. The multiplication is defined as

$$(\alpha_1, h_1) *_{L_{2,1}} (\alpha_2, h_2) = (\alpha_1 + \alpha_2^{h_1}, h_1 + h_2)$$

where $\alpha^h(x) = \alpha(x - h)$.

1. Show the following bound on the volume of the ball of radius n
$$|B(n)| \geq 2^{\frac{n}{2}}.$$
Hence $\limsup_n \sqrt[n]{B(n)} > 1$.
2. Show that the following sequence of sets
$$F_m = \{(\alpha, h) |\, supp(\alpha) \subset [-m, m],\, -m \leq h \leq m\}$$
is a Følner sequence.

4. Suppose a group G is nonamenable : its action on itself by right multiplication is paradoxical, i.e., there exists disjoint subests $A_1, ..., A_n, B_1, ..., B_m$ in G and elements $g_1, ..., g_n, h_1, ..., h_m$ in G such that

$$G = \cup_{i=1}^{n} g_i A_i = \cup_{j=1}^{m} h_j B_j.$$

The minimal number $m + n$ over all such decompositions for G is called the *Tarski number* of G, denoted $\mathcal{T}(G)$. The Tarski number of an amenable group is defined to be $+\infty$. Prove the following statements :

1. The Tarski number of any group is at least 4.
2. The Tarski number of the free group F_2 is 4.
3. If $H \leq G$, then $\mathcal{T}(G) \leq \mathcal{T}(H)$.
4. If N is a normal subgroup of G, then $\mathcal{T}(G) \leq \mathcal{T}(G/N)$.
5. A group G contains a subgroup isomorphic to F_2 if and only if $\mathcal{T}(G) = 4$.
6. * If G is a torsion group, then $\mathcal{T}(G) \geq 6$.
7. * If N is a normal amenable subgroup in G, then $\mathcal{T}(G) = \mathcal{T}(G/N)$.