Lecture 4. 4.12.20

From Probability & Dynamics to Group Theory.

Remark. Amenability for topological groups.

The original definition of von Neumann:

$\exists\, \mu: \mathcal{B}(G) \to [0,1]$ s.t. $\mu(G)=1$;
μ fin.-additive and G-left-invariant

is given for an abstract group.

Similarly: G amen $\Leftrightarrow$ $G \not\subset G$ non paradoxical

For locally compact groups with a Haar measure

fin. additive measures on G $\hookrightarrow$ invariant means in the following sense:

$m \in (L^\infty(G))^*$ s.t. $m(\mathbb{1}) = 1$;
$m(f) \geq 0$ if $f \geq 0$, $f \in L^\infty(G)$

$g \in G$, $f \in L^\infty(G)$ left G-invariant:

$m(g.f) = m(f)$ where $g.f(h) = f(g^{-1}.h)$

For locally compact groups, we can use the notion of being compactly generated instead of being finitely generated.

Følner criterion of amenability holds in this more general setting:

G is amenable iff $\forall A \subset G$ compact subset and $\forall \varepsilon > 0$ $\exists$ a Borel subset $F \subset G$ with positive finite Haar measure

st $\dfrac{\chi(aF \setminus F)}{\chi(F)} < \varepsilon \quad \forall a \in A.$

(Følner sequences $\rightsquigarrow$ Følner nets).

B. Bekka, P. de la Harpe, A. Valette. "Kazhdan's Property (T)".
Chapter 6: Amenability.

Many equivalent reformulations of amenability,
Fixed point property; $1_G \prec \lambda_G$
etc.

Kesten's Theorem for loc. compact groups

(Dermenic-Guivarc'h '73).

G loc. compact.
μ a Borel prob. on G $\rightsquigarrow$
$\forall \pi$ linear repr. of G in $\mathcal{L}(\mathcal{H})$,
define $\pi(\mu) \in \mathcal{L}(\mathcal{H})$ by

$$\langle \pi(\mu)\xi, \eta \rangle = \int_G \langle \pi(g)\xi, \eta \rangle \, d\mu(g)$$

$\forall \xi, \eta \in \mathcal{H}$.

Thm. Let μ be a prob. measure on G s.t. $\overline{\langle \mathrm{supp}\, \mu \rangle} = G$ and μ is abs. continuous wrt the Haar measure χ on G.

Then G is amenable iff $1 \in \mathrm{Sp}(\lambda_G(\mu))$.

Poisson-Furstenberg boundary.

About lie groups:

- For a local field K, the groups $SL_n K$, $n \geq 3$ and $Sp_{2n} K$, $n \geq 2$ have property (T).
- G loc. comp. has (T) & is amen $\Rightarrow$ G is compact.
- Compact groups, Abelian groups, Nilpotent groups, Solvable groups are amenable.
- Theory of approximate groups: Breuillard, Green, Tao ---

Lecture 4. 4.12.20

From Probability & Dynamics to Group Theory.

Reminder:

$EAG \overset{?}{=} AG \overset{?}{=} NF$ — Day's Question

No — Grigorchuk's group and other groups of intermediate growth

No — free Burnside groups of large exponent and some other examples of infinite f.g. torsion groups, like Tarski monsters.

- But the answer is yes in some "well-behaved" classes of fin. gen. groups, e.g. in linear groups.
- Maybe we can modify the definition of EAG and NF a bit?

Ideas:

1) Instead of EAG = the closure of finite gps ∪ Abelian gps under group extensions & direct limits

consider the closure of the class of all groups of subexponential growth under the same elementary operations =: SAG

subexponentially amenable gps.

SAG = AG ?

Answer: No.

$EAG \subsetneq SAG \subsetneq AG$

SAG ∌ Basilica group ∈ AG

Grigorchuk, Żuk

Bartholdi, Virág.

Basilica group comes from Complex Dynamics.

Let f be a complex rational function

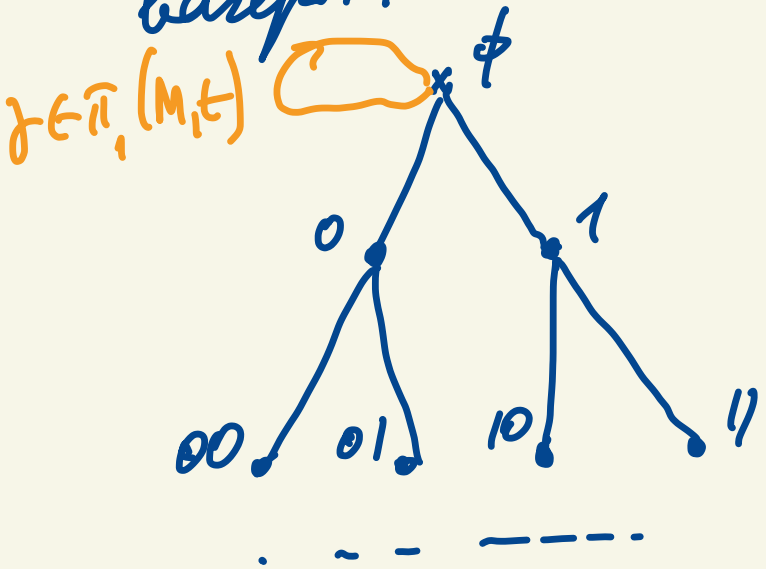
e.g. $f(z) = z^2 - 1$

$$f : \overbrace{\mathbb{C} \setminus f^{-1}(P_f)}^{M_1} \longrightarrow \overbrace{\mathbb{C} \setminus P_f}^{M}$$

P_f = postcritical set

We look at the tree T of preimages of a basepoint.

In our example, it is the infinite binary tree.



Preimages of γ under f define a permutation of vertices of each level of the tree. It's an automorphism of T.

Hence we get

$\varphi : \pi_1(M, t) \to Aut(T)$ homomorphism.

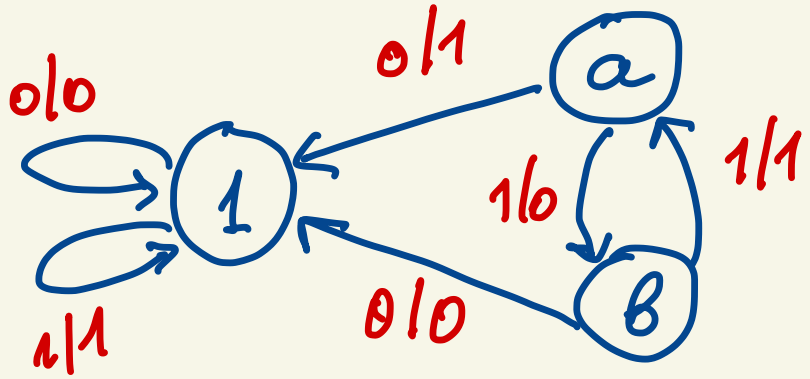
$\pi_1(M, t) / \ker \varphi$ is the iterated monodromy group of f.

$IMG(f) = \pi_1(M, t) / \ker \varphi$

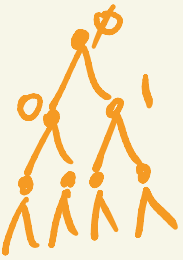
$IMG(f) \leq Aut(T)$.

Basilica group = $IMG(z^2-1) =: \mathcal{B}$.

$\mathcal{B} = \langle a, b \rangle$. Its action on the infinite binary tree is described by the following automaton:



A vertex at the n-th level of the tree is a binary word w of length n.
=
$w = 0w'$ or $1w'$



$a \cdot 0w' = 1w'$

$a \cdot 1w' = 0b \cdot w'$

$b \cdot 0w' = 0w'$

$b \cdot 1w' = 1a \cdot w'$

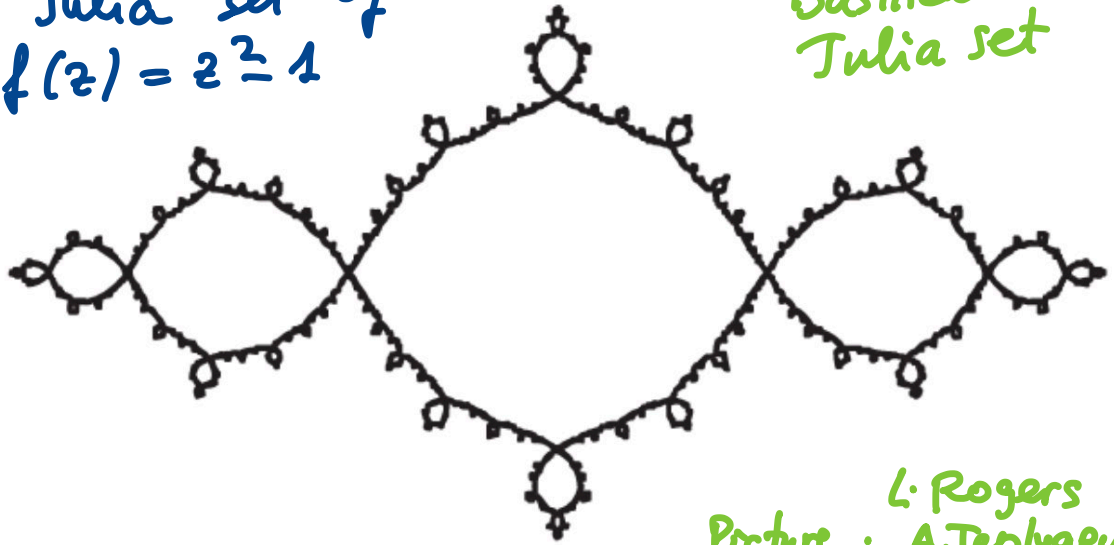
$\rightsquigarrow$ $a = \varepsilon(1, b)$
$b = (1, a)$

ε = non-trivial action at the root of the tree
EG $Sym(2) \neq e$

This is an example of an automata or self-similar group.

Julia set of $f(z) = z^2 - 1$

Basilica Julia set



L. Rogers
Picture : A. Teplyaev

$a \cdot 0w' = 1w'$

$a \cdot 1w' = 0b \cdot w'$

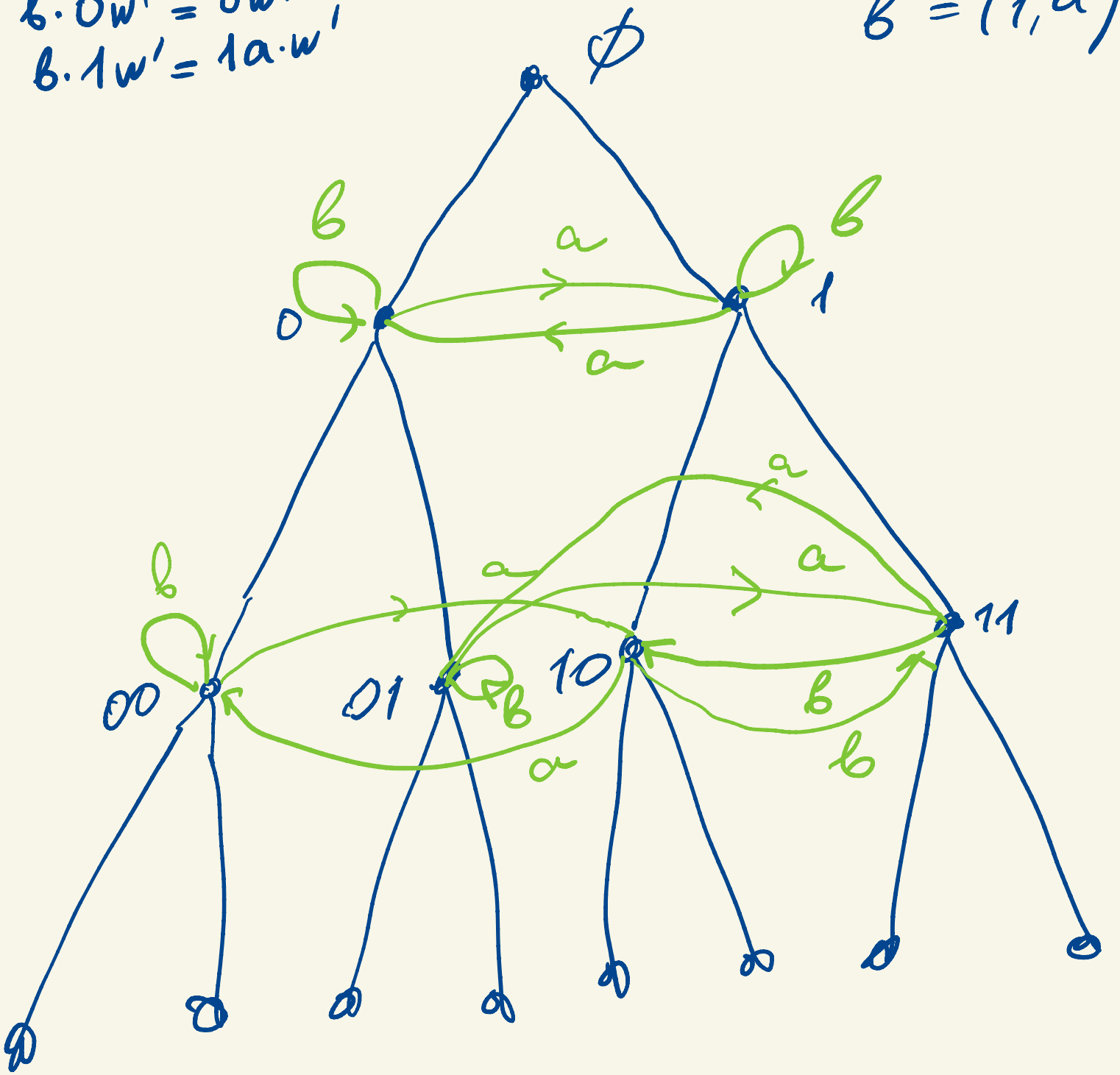
$b \cdot 0w' = 0w'$

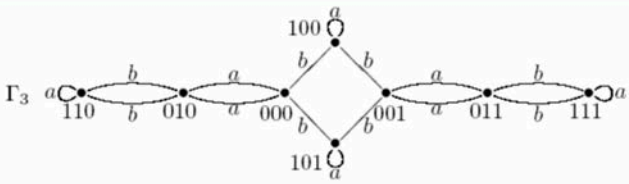
$b \cdot 1w' = 1a \cdot w'$

$B = \langle a, b \rangle$

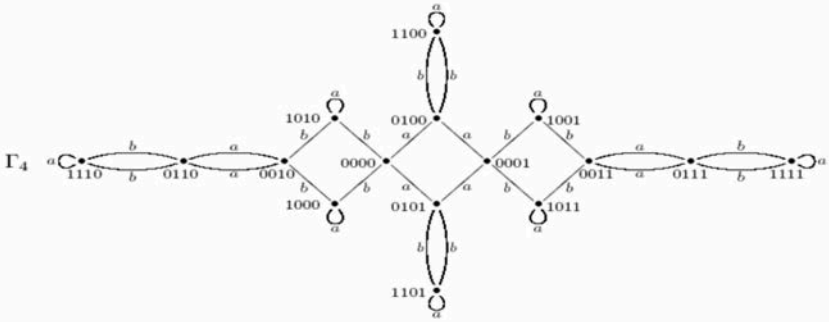
$a = \varepsilon(1, b)$

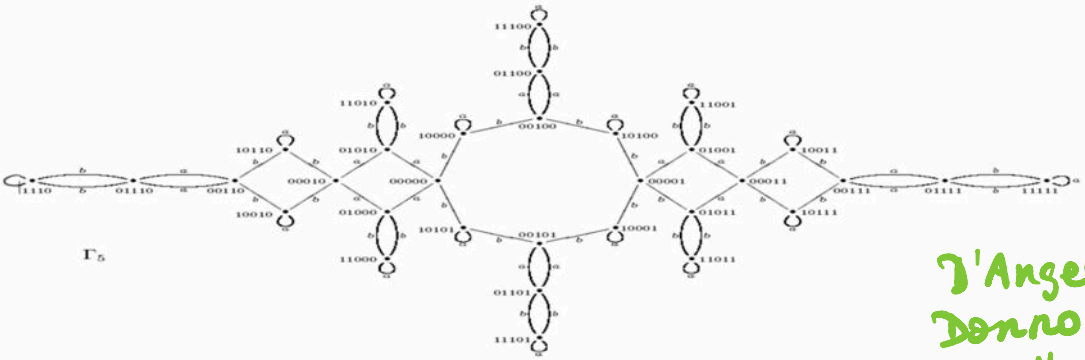
$b = (1, a)$





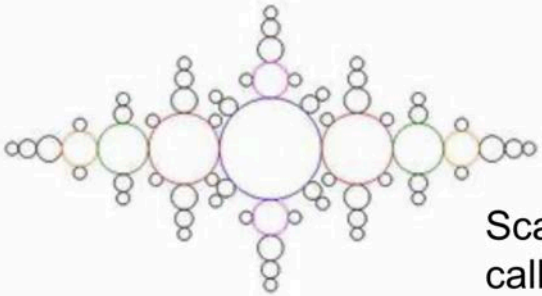
Schreier graphs of the action of the Basilica group on the levels of the binary tree





D'Angeli, Donno, 2010. Matter, N.

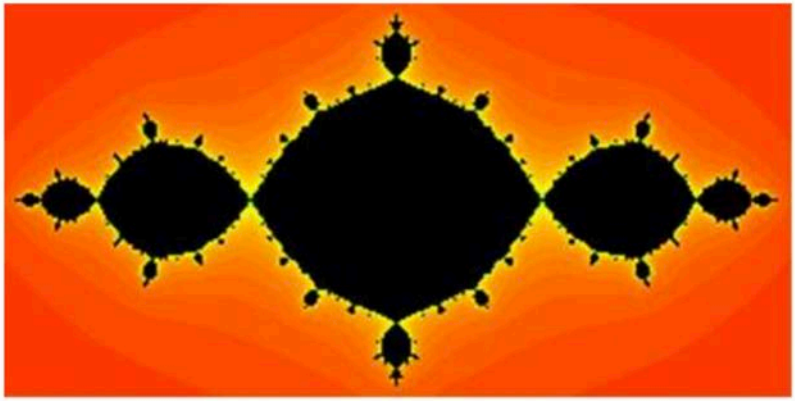
Γ_6



Scaling limit
called Limit space of G
(Nekrashevych)

$n \to \infty$ ...

B = IMG (z^2-1)



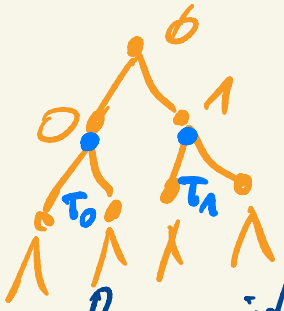
Julia
set of z^2-1

IMG of post-critically finite polynomials are amenable. In fact they are even Liouville.

(Note that basilica has exponential growth: the generators a, b generate a free subsemigroup.)

Let μ be a prob. on G and $(g_n)_{n}$ the corresponding random walk.

Write, for $g \in B$, $g = \sigma(g^{(0)}, g^{(1)})$ with $\sigma \in Sym(2)$, $g^{(0)}, g^{(1)} \in B$.



By iterating, we define $g^{(w)} \in G$ for each w binary word, projections of g on the subtree growing down from w: $T_w \simeq T$.

Look at the induced random walk μ^w at stopping times n s.t. $g_n w = w$.

$g_n T_w = T_w$.

They then show that for certain $w \in T$, μ^w is a convex combination of μ and δ_e (they call such measures self-similar). For such measures on G, the rate of escape of the RW vanishes:

$$\lim_n |g_n|/n = 0$$

which is equivalent to the Liouville property.

$$EAG \subsetneq SAG \subsetneq AG$$

Some other elementary classes?

2) $AG \subset NF$
$\neq$ no subgp $\simeq F_2$.

Idea: let us relax the condition NF.

Gaboriau, R. Lyons: Every [fin. gen.] (countable) non-amenable group "measurably contains" a free subgroup.

More precisely, by the Pak-N. result, $\forall$ nonamenable group has a Cayley graph Γ with $p_c < p_u$, so that $\forall p \in (p_c, p_u)$, there coexist in Γ infinitely many infinite clusters $\mathbb{P}_p$-a.s.

$G \curvearrowright (\{0,1\}^{E(\Gamma)}, \mathbb{P}_p)$ free pmp ergodic action.

Claim: $\exists$ a free pmp action of F_2 on the same space, s.t. $F_2 \cdot x \subset G \cdot x$ for a.e. x.

The orbits of the F_2-action are constructed from the infinite clusters, using the fact that these clusters are subgraphs in Γ and hence are connected graphs of degree $\leq |S|$.

(we assume for simplicity $|S| = 2k$). $S = S^{-1}$

Each such graph can transformed into a Schreier graph of the free group by completing the "incomplete" vertices (of deg $< 2k$) by gluing branches of the $2k$-regular tree.

J. Stallings. "Topology of finite graphs." 1983.

$H < G = \langle S \rangle \rightsquigarrow \Gamma = Sch(G, H, S)$.

$V = G/H \qquad E = \{(Hg, Hg \cdot s)\}_{\substack{g \in G \\ s \in S}}$

$$F_k = \langle \overbrace{a_1, \dots a_k}^{= A} \rangle$$

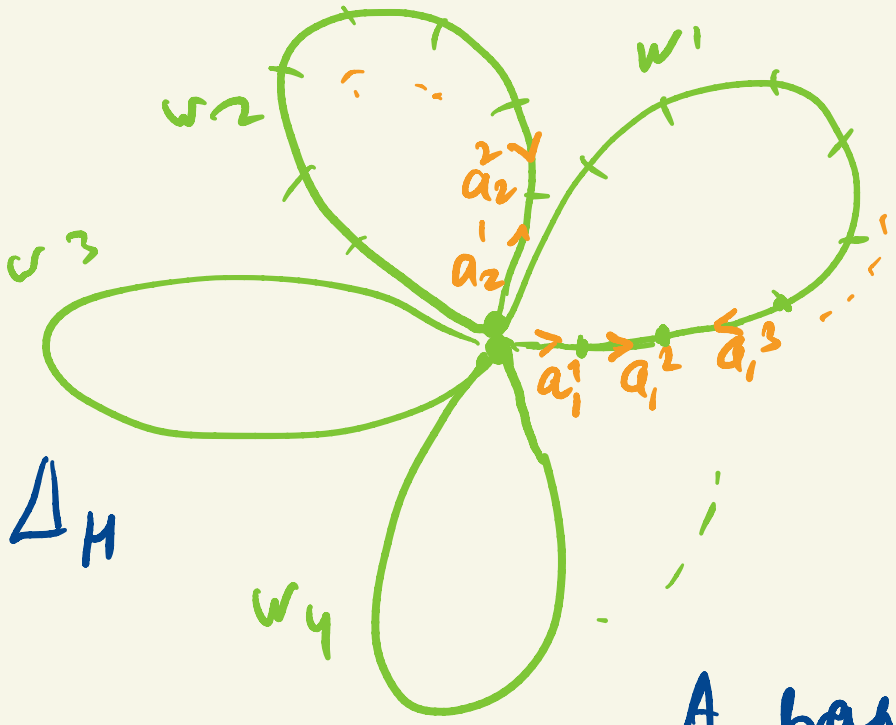
Grigorchuk, Kaimanovich, N. 2012

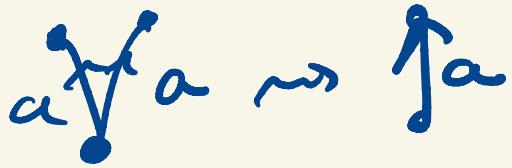
$$F_k \geq H = \langle w_1, w_2, w_3 \dots \rangle$$

$$w_i = w_i(A \cup A^{-1})$$

Stallings foldings:

H is a free group





$\Delta_H / \sim$ = Core H

foldings

A graph of degree $\leq 2k$.

Complete where needed by boundary of T_k.

$\rightsquigarrow$ $\Gamma = Sch(F_k, H, A \cup A^{-1})$.

Remark: A geometric version of von-Neumann's Pbm also gets a positive solution:

K. Whyte ('99). G nonamenable $\Rightarrow$ every Cayley graph can be cut into pieces uniformly bilipschitz equivalent to the regular tree of degree $|S|$.

If $G > F_2$, then these pieces can be chosen to be cosets of F_2.

Gaboriau-Lyons construction of a "measurable" subgroup F_2 in any non-amenable group led to progress in one more open conjecture on hyperfiniteness.

Dixmier Non-Unitarizability Problem.

Inspired by the classical result of Stökefalvi-Nagy '47.

Remark. If an operator $A \in \mathcal{L}(\mathcal{H})$ is conjugated to a unitary operator, then it is uniformly bounded:

$$\sup_{n \in \mathbb{Z}} \|A^n\| < \infty.$$

Thm. The converse is true.

Dixmier Non-Unitarizability Problem

Dixmier '50s: G amenable $\Rightarrow$ $\forall$ representation $\pi: G \to \mathcal{L}(\mathcal{H})$ that is uniformly bounded, i.e. $\exists C > 0: \|\pi(g)\| < C$ $\forall g \in G$, can be made unitary by conjugation by an invertible operator S: $S\pi(g)S^{-1}$ is unitary for $\forall g \in G$.

Is the converse true?

- F_2 is not unitarizable: $\exists$ concrete examples of unif. bnded, not unitarizable representations: Mantera-Zappa '83
Pytlik-Schwarz '86
Bożejko '87

Pisier '05.

- Unitarizability passes to subgroups

$\Rightarrow \forall G > F_2$ is non-unitarizable.

for groups without free subgroups.

Open Problem: Is every non-amenable group non-unitarizable? "Dixmier Pbm."

Monod: ICM talk '06.

Monod-Epstein '09: first examples of non-unitarizable groups without free subgroup. (Using "monsters" constructed by D. Osin)

Monod, Ozawa '09: $B(m,n)$ are non unitarizable for certain $n \gg 665$.