Three Surface Theorems

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Analysis, Probability, Mathematical Physics

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Outline

Almgren's Frequency Function

- Harmonic Function: $h : B_R \subset \mathbb{R}^n \to \mathbb{R}$, $\Delta h = \text{div}(\text{grad}\, h) = 0$

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Theorem

Frequency: $N(r) = \frac{rH'(r)}{H(r)}$ *is increasing*

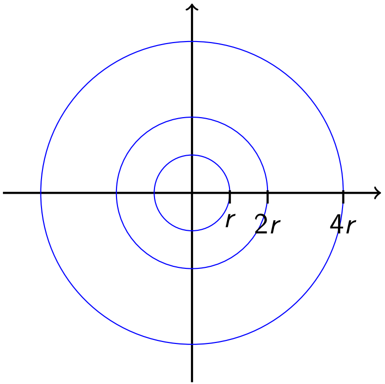
Convexity: $\log(H(e^t))$ *is convex, or equivalently*

$$\frac{H''(r)\,H(r) - H'(r)^2 + \frac{1}{r}H(r)\,H'(r)}{H(r)^2} \geq 0.$$

Midpoint Convexity: $H(2r) \leq \sqrt{H(r)H(4r)}$

$$\log(H(e^t))' = N(e^t)$$

Implications



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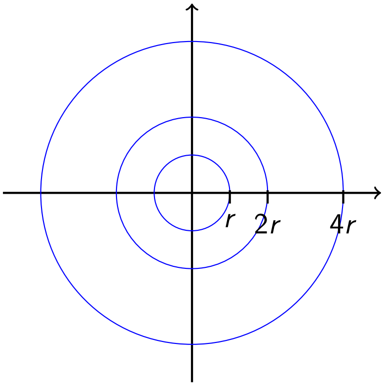

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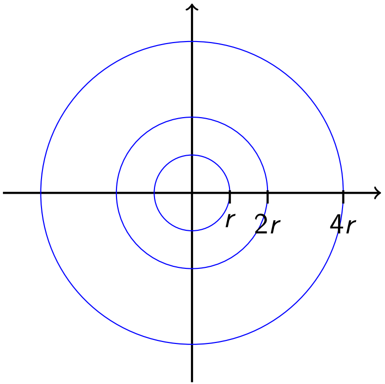

- Propagation of smallness.
- Unique continuation, and controlling the order of vanishing.
- $N(r)$ controls the vanishing order of h.

Implications (Continued)

Example (Homogeneous harmonic polynomials)

Let h be a homogeneous harmonic polynomial of degree d

- $N(r) = \frac{rH'(r)}{H(r)} = 2d + n - 1$
- $H(2r) = \sqrt{H(r)H(4r)}$

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Polynomial growth

Let h be a harmonic function and $s < r$ then

$$\left(\frac{r}{s}\right)^{N(s)} \leq \frac{H(r)}{H(s)} \leq \left(\frac{r}{s}\right)^{N(r)}$$

Extension to Eigenfunctions

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Theorem (Midpoint Theorem)

For all u there exists $C, c > 0$ such that

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Proof.

Using the inequality for the harmonic extension functions. □

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The result is optimal, in the sense that we can not change Ce^{ckr} to something with slower growth.

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Other results

Elliptic Equation: Garofalo and Lin 86

Schrödinger Equation: Kukavica 98

On Riemannian Manifolds

Riemannian Manifold: $(M, \mathbf{g})$

Spherical Coordinates: $p \in M$ and $r(x) = \text{dist}(x, p)$, and S_r is the sphere with radius r

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- When taking the derivative; some change comes from the function h, and from the area

$$H'(r) = 2 \int_{S_r} h h_r \, dS + \int_{S_r} h^2 \Delta(r) \, dS$$

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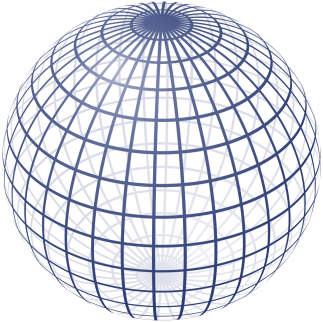
$$H''(r) = 2 \int_{S_r} h_r^2 \, dS + 2 \int_{S_r} hh_{rr}^2 \, dS + 4 \int_{S_r} hh_r \Delta(r) \, dS + \int_{S_r} h^2 (\Delta(r))^2 \, dS + \int_{S_r} h^2 \Delta(r)_r \, dS$$

Controlling the Derivative of S_r



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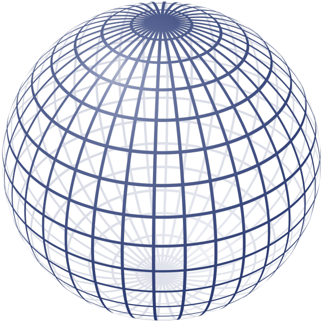

$$\cot_K(t) = \begin{cases} \dfrac{\sqrt{K}\cos(t\sqrt{K})}{\sin(t\sqrt{K})}, & \text{when } K > 0 \\ \dfrac{1}{t}, & \text{when } K = 0 \\ \dfrac{\sqrt{-K}\cosh(t\sqrt{-K})}{\sinh(t\sqrt{-K})}, & \text{when } K < 0 \end{cases}$$

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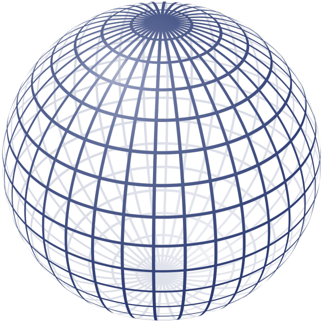

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$$\partial_r(\Delta r) = -\operatorname{Ric}(\partial_r, \partial_r) - |\nabla^2 r|^2$$

The Result for Riemannian Manifolds

Theorem (Rauch Comparison Theorem)

Assume that the scalar curvature of M satisfy $\kappa\|X\|^2 \leq \operatorname{Sec}(X, X) \leq K\|X\|^2$. *Then*

$$\cot_K(r) g_{S_r} \leq \nabla^2 r \leq \cot_\kappa(r) g_{S_r}$$

which implies that

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Theorem (Mangoubi 13, slightly improved for $K > 0$ in B. 19)

$$\begin{aligned}(\log H(t))'' + (\cot_K(t) + (n+1)(\cot_\kappa(t) - \cot_K(t)))(\log H(t))' \\ \geq -(n-1)K + (n-2)\min(K, 0) - (n-1)(K-\kappa)\end{aligned}$$

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- Which we can integrate and get a convexity inequality

Other Families of Surfaces

$$f : U \subset (M, \mathbf{g}) \to [0, \infty)$$

be a smooth regular function at every point except possibly 0.

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Other Families of Surfaces (Continued)

Theorem (Hörmander unpublished work)

There exists a $\mathcal{K}$ only depending on f such that for any harmonic function h

$$\int_{S_t} \frac{\left|\operatorname{grad}_{S_t} h\right|^2 - h_n^2}{\left|\operatorname{grad} f\right|} \, dS \geq -\mathcal{K}(t) \int_{S_t} \left|\operatorname{grad} h\right|^2 \, d\,\mathrm{vol} \, .$$

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Theorem (B. 19)

Assume that

$$m(t) \leq \frac{\Delta f}{|\text{grad } f|^2} \leq M(t) , \text{ and } \left\langle \text{grad} \left(\frac{\Delta f}{|\text{grad } f|^2} \right) , \frac{\text{grad } f}{|\text{grad } f|^2} \right\rangle \geq g(t)$$

on S_t *and assume that* $\mathcal{K}(t) + M(t) \geq 0$*. Then* H *satisfies*

$$(\log H(t))'' + (\mathcal{K}(t) + M(t))(\log H(t))' \geq g(t) + m(t) M(t) + m(t) \mathcal{K}(t) .$$

Example: Ellipse

$$D = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{bmatrix}, \quad \text{with } 0 < a_i.$$

The level surfaces of $f(x) = \sqrt{\langle x, Dx \rangle}$.

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The level surfaces of $f(x) = \sqrt{\langle x, Dx \rangle}$.

$$\frac{H''(t)\,H(t) - H'(t)^2 + \frac{C_1}{t} H'(t)\,H(t)}{H(t)^2} \geq -C_2/t^2,$$

where C_1, C_2 only depends on D.

Example: Torii

Let $S^k \times 0^{n-k-1} \subset \mathbb{R}^n$. Define f to be

$$f(x) = \sqrt{(r_{k+1}(x) - 1)^2 + x_{k+2}^2 + \cdots + x_n^2},$$

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The function f is smooth if $f < 1$, hence we will assume that $f \leq 1 - \varepsilon$. In this case we get the same inequality with

$$C_1 = 1 + k\left(\frac{1}{\varepsilon} - \frac{1}{2-\varepsilon}\right) \quad \text{and} \quad C_2 = k\left(\frac{(2n-3)(1-\varepsilon)}{\varepsilon(2-\varepsilon)} - \frac{2}{\varepsilon^2}\right).$$

Thank you for listening!