Non-existence of a universal zero entropy system for amenable group actions

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Amenability and invariant measures

A countable G is called *amenable* if it satisfies the Følner condition. That is, there exists a sequence of finite subsets $F_n \subset G$ such that for any $g \in G$

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

The following properties are equivalent.

1. Any (left) continuous action of G on a compact metric space X admits an invariant probability measure.
2. G is amenable.

Thus, the set $M_G(X)$ of all invariant measures of a topological action $G \curvearrowright X$ is always non-empty.

Question

What can be the set of ergodic measures on X?

The variational principle

There are two well–known characteristics of G–actions.

- The entropy $h_{top}(X, G)$ of a topological action.
- The entropy $h(X, \mu, G)$ of a measure–theoretic action.

They are related in the following way.

Theorem (The variational principle)

Let $G \curvearrowright X$ be a topological G–action. Then

$$h_{top}(X, G) = \sup_{\mu \in M_G(X)} h(X, \mu, G).$$

In particular, if the topological entropy is zero, then the measure theoretical entropy is zero as well.

Universal systems

Definition

A topological system (X, G) is called *universal* for some class $\mathcal{S}$ consisting of ergodic $G-$actions if

1. for any ergodic $\mu \in M_G(X)$ the system (X, μ, G) belongs to $\mathcal{S}$;
2. for any system $(Y, \nu, G) \in \mathcal{S}$ there exists $\mu \in M_G(X)$ such that (X, μ, G) and (Y, ν, G) are isomorphic.

Example

- The standard shift on $[0,1]^G$ is universal for the class of all $G-$actions.
- (Krieger'70, Seward'18) Bernoulli shift on $\{1, \ldots, n\}^G$ is universal for all actions with $h < \log n$ (and one more*).
- (Downarowicz, Serafin'16) There exists a universal $\mathbb{Z}-$system for $h \in [0, \alpha)$ or $h \in [0, \alpha]$, $\alpha > 0$.

Universal zero entropy system

Question (B. Weiss)

Does there exist a system (X, G) with is universal for the class of all ergodic measure–preserving actions of zero entropy?

For the case of $\mathbb{Z}$, the negative answer was given by J. Serafin ('13). However, this question is still open for general amenable groups. Our main result is the following theorem.

Theorem (G.V. '20)

Let $G \curvearrowright X$ be a continuous action of a non–periodic amenable group. Assume that for any ergodic zero entropy system (Y, ν, G) there exists $\mu \in M_G(X)$ such that

$$(X, \mu, G) \cong (Y, \nu, G).$$

Then the topological entropy of (X, G) is positive.

Scaling entropy invariant

The main tool we implement in the proof is the notion of *scaling entropy* proposed by A.M.Vershik.

- Let G be a countable group. We call a fixed sequence of finite subsets $\lambda = \{F_n\}$ *equipment* of the group.
- We require the equipment to be *suitable*.

Example

- $G = \mathbb{Z}$, $F_n = \{0, \ldots, n-1\}$;
- Any equipment of an abelian group is suitable;
- An amenable group with a Følner sequence;
- Free group equipped with the standard balls is suitable.

Admissible semimetrics

Let $\rho\colon (X^2, \mu^2) \to [0, +\infty)$ be a non-negative symmetric measurable function which satisfies the triangle inequality.

Example

Let ξ be a measurable partition. The corresponding cut semimetric $\rho_\xi(x, y) = 0$ if x and y lie in the same cell of ξ and $\rho_\xi(x, y) = 1$ otherwise.

Define the ε*–entropy* $\mathbb{H}_\varepsilon(X, \mu, \rho)$ of ρ as follows.

Definition

Let k be the minimal integer such that $X = X_0 \cup X_1$, where $\mu(X_0) < \varepsilon$ and the cardinality of the minimal ε–net in X_1 equals to k. Put

$$\mathbb{H}_\varepsilon(X, \mu, \rho) = \log_2 k.$$

If there is no such k put $\mathbb{H}_\varepsilon(X, \mu, \rho) = +\infty$

Admissible semimetrics

The following conditions are equivalent (Vershik, Petrov, Zatitskiy'16)

1. The semimetric ρ is separable on a subset of full measure.
2. $\mathbb{H}_\varepsilon(X, \mu, \rho)$ is finite for all $\varepsilon > 0$.

Definition

In this case, the semimetric ρ is called *admissible*.

These properties are evident.

- If ρ is admissible then all its shifts $g^{-1}\rho(x, y) = \rho(gx, gy)$, $g \in G$, are admissible as well.
- A finite averaging of an admissible semimetric is admissible

$$G^n_{av}\rho(x, y) = \frac{1}{|F_n|}\sum_{g\in F_n} \rho(gx, gy), \qquad x, y \in X.$$

Scaling entropy of an equipped group

Definition

- We say that $\Phi(n,\varepsilon) \preceq \Psi(n,\varepsilon)$ if $\forall \varepsilon\ \exists \delta\ \Phi(n,\varepsilon) \lesssim \Psi(n,\delta)$.
- We call two functions equivalent if $\Phi \preceq \Psi$ and $\Psi \preceq \Phi$.
- The equivalence class of Φ is denoted by $[\Phi]$.

We are ready to define the scaling entropy of a measure–preserving action of an equipped group.

1. Consider the function $\Phi_\rho(n,\varepsilon) = \mathbb{H}_\varepsilon(X,\mu,G^n_{av}\rho)$.
2. Take the corresponding equivalence class $\mathcal{H} = \mathcal{H}(X,\mu,G,\lambda) = [\Phi_\rho]$.

Theorem (Zatitskiy '15)

Let $(G,\lambda) \curvearrowright (X,\mu)$ be a measure–preserving action of a countable suitably equipped group. Assume that ρ and ω are admissible generating summable semimetrics. Then $[\Phi_\rho] = [\Phi_\omega]$.

Zero entropy

Zero entropy systems can be easily distinguished by means of scaling entropy.

Theorem

1. *Assume that* $h(X, \mu, G) > 0$ *then for any* $\Phi \in \mathcal{H}$ *and sufficiently small* ε
$$\Phi(n, \varepsilon) \asymp |F_n|.$$
2. *Let* $h(X, \mu, G) = 0$ *and* $\Phi \in \mathcal{H}$. *Then for all* ε
$$\Phi(n, \varepsilon) = o(|F_n|).$$

- Thus, it is always true that
$$\Phi(n, \varepsilon) \lesssim |F_n|.$$
- We will looking for systems for which *the equivalence is almost achieved*.

Actions of almost complete growth

Definition

We say that (G, λ) *admits actions of almost complete growth* if for any non−negative function $\phi(n) = o(|F_n|)$ there exists a measure−preserving system (X, μ, G) such that for any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$

- $\Phi(n, \varepsilon) = o(|F_n|)$,
- $\Phi(n, \varepsilon) \not\lesssim \phi(n)$.

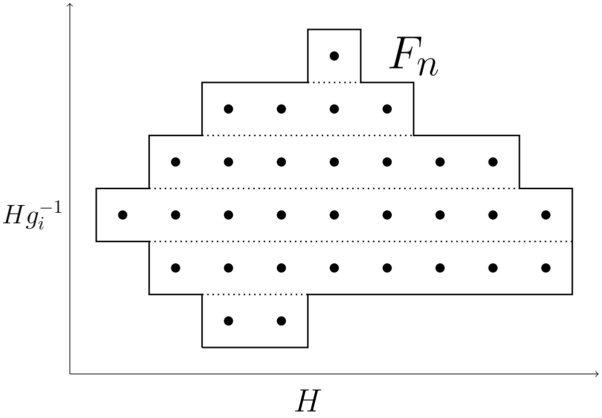
The first condition is equivalent to say that the measure entropy is zero.

1. We prove that the main theorem holds for any such group.
2. The main part: every non−periodic amenable group admits ergodic actions of almost complete growth with respect to arbitrary Følner equipment.
   - Construct almost complete actions for the group $\mathbb{Z}$.
   - Apply coinduction from a subgroup to the whole G.

Step I: Sharpening Følner sets

Let $H = \langle h \rangle$ — be a subgroup generated by an element h of infinite order and F_n be a (h, ε^2)–invariant Følner set. There is $W_n \subset F_n$ such that

- All the sections $S_n^i = H \cap W_n g_i$ are (h, ε)–invariant.
- The asymptotic relation $|W_n \triangle F_n| = o(|F_n|)$ holds.



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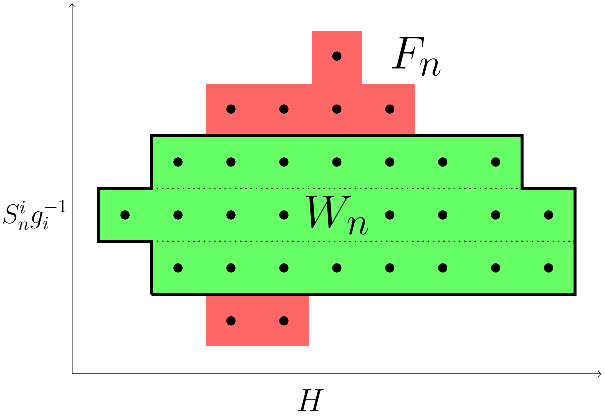

Step II: Adic transformation

Now the construction of almost complete actions begins.

Step II

We construct a special $\mathbb{Z}$–actions satisfying uniform estimates for the scaling entropy over the given system of subsets $\{S_n^i\}$

The desired examples can be obtained by the *adic action* (Vershik's automorphism) on the graph of ordered pairs endowed with certain central measures.

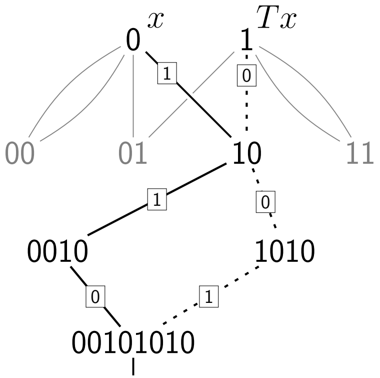

Figure: Vershik's automorphism

Step III: Coinduced actions

Finally, we apply the coinduction procedure from the subgroup $H \cong \mathbb{Z}$ to the whole group G.

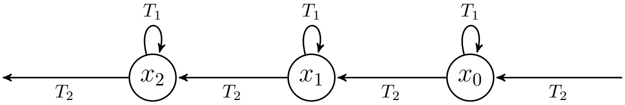

Figure: $\mathrm{CInd}_{\mathbb{Z}}^{\mathbb{Z}^2} T_1$

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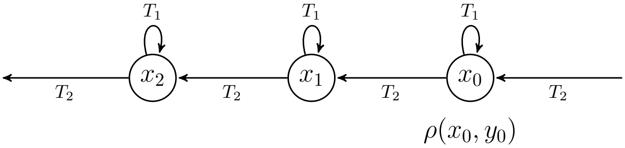

Figure: Semimetric ρ

- Choose a semimetric ρ depending only on the x_0.

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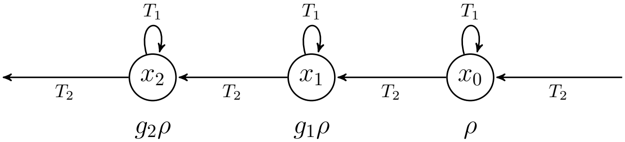

Figure: Semimetric ρ and its shifts

- Choose a semimetric ρ depending only on the x_0.
- Any shift $g_i s\rho$, $s \in H$, also depends only on one component.

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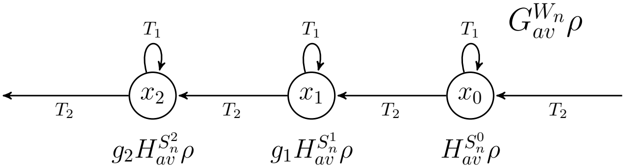

Figure: Averaging of ρ over W_n

- Choose a semimetric ρ depending only on the x_0.
- Any shift $g_i s\rho$, $s \in H$, also depends only on one component.
- The averaging $G_{av}^{W_n}\rho$ decomposes into a weighted combination of $S_n^i\rho$.

Step IV: Lower bound for ε–entropy

The main technical statement that we need to complete the proof is the following lemma.

Lemma

Consider a family of admissible semimetric triples (X_i, μ_i, ρ_i). *Let* $\phi^{-1} s_i < \mathbb{H}_{4\varepsilon}(X_i, \mu_i, \rho_i) < s_i$, $s = s_1 + \ldots + s_k$. *Let* ρ *be the averaged semimetric:* $\rho = \frac{1}{s}\sum_{i=1}^{k} s_i \rho_i$. *Then*

$$\mathbb{H}_{\varepsilon^4}(X, \mu, \rho) \geqslant \frac{1}{\phi}\varepsilon^3 \sum_{i=1}^{k} \mathbb{H}_{4\varepsilon}(X_i, \mu_i, \rho_i) - k - 1,$$

where (X, μ) *is the product of* (X_i, μ_i).

This lemma implies that the sequence $\mathbb{H}_{\varepsilon^4}(X, \mu, G_{av}^{W_n}\rho)$ can not be bounded by any given one.

Thank you for your attention!