Cellular $\mathbb{A}^1$-homology of schemes associated with the Bruhat decomposition

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Motivation: Motivic version of Matsumoto's theorem

Computing $\pi_1^{\mathbb{A}^1}(G) = H_1^{\mathbb{A}^1}(G)$.

k: perfect field

$Ab(k)$: Nisnevich sheaves of abelian groups on $Sm/_k$.

$Ab_{\mathbb{A}^1}(k)$: Strictly $\mathbb{A}^1$-invariant sheaves

($H^n_{Nis}(-, M)$ is $\mathbb{A}^1$-invariant on $Sm/_k$ $\forall\ n \geq 0$)

$$\pi_n^{\mathbb{A}^1}(\mathcal{X}, \cdot) := a_{Nis}\left(U \longmapsto [S^n \wedge U_+, \mathcal{X}]_{\mathbb{A}^1, \cdot}\right) \in Ab_{\mathbb{A}^1}(k) \quad \forall\ n \geq 1.$$

$$H_n^{\mathbb{A}^1}(\mathcal{X}) := H_n(L_{\mathbb{A}^1} C_*(\mathcal{X}))$$

normalized chain complex

$\mathbb{A}^1$-localization functor

$L_{\mathbb{A}^1} : D(Ab(k)) \longrightarrow D_{\mathbb{A}^1}(k)$

subcat. of $D(Ab(k))$ of $\mathbb{A}^1$-local objects

<u>Morel</u>: $\tilde{H}_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge n}) = \underline{K}_n^{MW}$

e.g. $H_1^{\mathbb{A}^1}(SL_2) = H_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) = H_1^{\mathbb{A}^1}(S^1 \wedge \mathbb{G}_m \wedge \mathbb{G}_m)$

$= \tilde{H}_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge 2}) = \underline{K}_2^{MW}$.

Cellular $\mathbb{A}^1$-homology

<u>Definition</u>:

$X \in Sm/_k$ is cohomologically trivial if $H^n_{Nis}(X, M) = 0$ $\forall\ n \geq 1$ $\forall\ M \in Ab_{\mathbb{A}^1}(k)$.

Example: $\mathbb{A}^n$, $\mathbb{G}_m$, split tori.

Lemma:

If $X \in Sm/_k$ is affine and cohomologically trivial, then every vector bundle on X is trivial.

Proof:

Rank r vector bundle on $X \rightsquigarrow \xi \in [X, BGL_r]_{\mathbb{A}^1}$

Postnikov tower argument

$\Rightarrow$ obstructions to lifting ξ to $[X, BGL_{r-1}]_{\mathbb{A}^1}$

lie in $H^*(X, \pi_\bullet^{\mathbb{A}^1}(\text{fiber})(\mathcal{L})) = 0$

strictly $\mathbb{A}^1$-inv. line bundle twist

X : coh. triv. $\Rightarrow H^1(X, \mathbb{G}_m) = 0$, $H^n(X, M) = 0 \;\forall M \in Ab_{\mathbb{A}^1}(k)$.

$\square$

Definition:

A rank r vector bundle ξ on $X \in Sm/_k$ is (strictly) orientable if $\Lambda^r \xi$ is trivial.

(Strict) orientation of ξ : isomorphism $\theta : \mathbb{A}^1_X \to \Lambda^r \xi$.

Lemma:

Let X be cohomologically trivial and let ξ be a (trivial) rank r vector bundle on X. The induced pointed $\mathbb{A}^1$-homotopy class of

$$Th(\xi) \longrightarrow \mathbb{A}^r / \mathbb{A}^r \setminus \{0\} \wedge X_+$$

$$Th(\xi) = \xi / \xi - \{0\text{-section}\}$$

(which is an $\mathbb{A}^1$-weak equivalence) only depends on the chosen orientation of ξ.

Definition:

$X \in Sm/_k$ is called cellular if it admits an increasing filtration by open subschemes

$$\emptyset = \Omega_{-1} \subset \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_\ell = X$$

where for each i, $\Omega_i - \Omega_{i-1} \overset{\text{smooth}}{\underset{j_i}{\hookrightarrow}} \Omega_i$ is of codim i and every irreducible component of $\Omega_i - \Omega_{i-1}$ is smooth, affine and cohomologically trivial.

Motivic homotopy purity $\Rightarrow$

$$\Omega_i/\Omega_{i-1} \xrightarrow{\simeq_{\mathbb{A}^1}} Th(\nu_i),$$

where ν_i = normal bundle of j_i.

Cellular $\mathbb{A}^1$-chain complex of X:

$$\begin{tikzcd}[column sep=small] C_*^{cell}(X) : & \cdots \arrow[r] & H_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) \arrow[rr, "\partial"] \arrow[dr] & & H_{i-1}^{\mathbb{A}^1}(\Omega_{i-1}/\Omega_{i-2}) \arrow[r] & \cdots \\ & & & H_{i-1}^{\mathbb{A}^1}(\Omega_{i-1}) \arrow[ur] & & \end{tikzcd}$$

$$C_*^{cell}(X) \in D(Ab_{\mathbb{A}^1}(k))$$

Cellular $\mathbb{A}^1$-homology of X:

$$H_n^{cell}(X) := H_n(C_*^{cell}(X)).$$

Remarks:

1) $H_n^{cell}(X) = 0$, $n < 0$, $n > \dim X$

ℓ has "nothing to do" with $\dim X$.

2) X is $\mathbb{A}^1$-$(n-1)$-connected $\Rightarrow$ $H_0^{cell}(X) = \mathbb{Z}$,

$$H_i^{cell}(X) = 0 \quad \text{for } 1 \leq i \leq n-1 ,$$
$$H_n^{\mathbb{A}^1}(X) \xrightarrow{\simeq} H_n^{cell}(X) ,$$
$$H_{n+1}^{\mathbb{A}^1}(X) \twoheadrightarrow H_n^{cell}(X) .$$

3) After choosing a trivialization of v_i,
$$\begin{aligned} H_i^{\mathbb{A}^1}(\Omega_i/\Omega_{i-1}) &= H_i^{\mathbb{A}^1}(Th(v_i)) \\ &= H_i^{\mathbb{A}^1}(S^i \wedge \mathbb{G}_m^{\wedge i} \wedge (\Omega_i - \Omega_{i-1})_+) \\ &= \tilde{H}_0^{\mathbb{A}^1}(\mathbb{G}_m^{\wedge i}) \otimes H_0^{\mathbb{A}^1}(\Omega_i - \Omega_{i-1}) \\ &= \underline{K}_i^{MW} \otimes H_0^{\mathbb{A}^1}(\Omega_i - \Omega_{i-1}) . \end{aligned}$$

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G: split, semisimple, almost simple, simply connected algebraic group $/k$.

Fix a <u>pinning</u> of G:

- maximal k-split torus T
- Borel subgroup $B \supset T$
- an isomorphism $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha \quad \forall \alpha \in \Delta$.

(first two) determines Δ: basis of simple roots

Weyl group of G: $W = N_G T / T$.
(generated by simple reflections s_α, $\alpha \in \Delta$)

<u>Convention:</u> For all $x \in k^\times$,
$$w_\alpha(x) = u_\alpha(x) \cdot u_{-\alpha}(x^{-1}) \cdot u_\alpha(x) \in N_G T$$
lifts $s_\alpha \in W$. Set $\boxed{\dot{s}_\alpha = w_\alpha(-1)} \quad \forall \alpha \in \Delta$.

Let $w_0 \in W$ be the longest element of W
$\ell = \text{length}(w_0)$.

Bruhat cellular structure on G:

$$\underset{\underset{B\dot{w}_0 B}{\parallel}}{\Omega_0} \subset \Omega_1 \subset \cdots \subset \Omega_\ell = G$$

(normal red. exp.)

For every $w \in W$, choose a reduced expression and set

$$U_w := \prod_{\alpha \in \Phi_w} U_\alpha$$

where $\Phi_w := \{ \alpha \in \underset{\text{+ve roots}}{\Phi^+} \mid w(\alpha) \in \Phi^- \}$.

$$U_{w^{-1}} \times \dot{w} T \times U \xrightarrow{\simeq} B\dot{w}B$$

cohomologically trivial

$$\Omega_i - \Omega_{i-1} = \coprod_{\substack{w \in W \\ \ell(w) = \ell - i}} B\dot{w}B .$$

$C_*^{cell}(G)$: cellular $\mathbb{A}^1$-chain complex of G

$$\boxed{C_i^{cell}(G) = \bigoplus_{\substack{w \in W \\ \ell(w) = \ell - i}} \underline{K}_i^{MW} \otimes H_0^{\mathbb{A}^1}(B\dot{w}B)}$$

For every $w \in W$, $U_w U_{w_0 w} \xrightarrow{\simeq} U$

Set $v := w_0 w$.

Then

$$\dot{v}\Omega_0 = \prod_{\substack{\beta \in \Phi^-, \\ w^{-1}(\beta) \in \Phi^+}} U_\beta \cdot U_{w^{-1}} \cdot \dot{w} T U$$

$$= \prod_{\beta \in -\Phi_{v^{-1}}} U_\beta \cdot B\dot{w}B \supset B\dot{w}B$$

Choose a (horizontal) reduced expression $\tilde{w}$ of each $w \in W$. Then

$$C_i^{cell}(G) \simeq \bigoplus_{\substack{w \in W \\ \ell(w) = \ell - i}} \underline{K}_i^{MW} \otimes \mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega} T]$$

$$\mathrel{\rotatebox{90}{$\cong$}}$$

$$\bigoplus_{-} K_i^{MW} \otimes \mathbb{Z}_{\mathbb{A}^1}[T]$$

<u>Aim</u>: To compute $H_1^{cell}(G)$.

Need to understand
- structure of $\mathbb{Z}_{\mathbb{A}^1}[T]$
- differentials in $C_*^{cell}(G)$.

<u>Structure of $\mathbb{Z}_{\mathbb{A}^1}[T]$:</u>

Choose $\prod_{i=1}^r \alpha_i^\vee : \prod_{i=1}^r \mathbb{G}_m \xrightarrow{\simeq} T$

(G simply connected $\Rightarrow$ $\alpha_i^\vee$: simple coroots)

F/k : field extn. $u \in F^\times \rightsquigarrow (u) \in K_1^{MW}(F)$

$t \in T \rightsquigarrow [t] \in \mathbb{Z}_{\mathbb{A}^1}[T]$ $\quad$ T: pointed at 1

$(t) := [t] - [1]$.

$\alpha_i^\vee : \mathbb{G}_m \to T$ induces $\underset{\substack{\| \\ K_1^{MW}}}{\mathbb{Z}_{\mathbb{A}^1}(\mathbb{G}_m)} \longrightarrow \mathbb{Z}_{\mathbb{A}^1}[T]$

$u \longmapsto (u)_i$

$:= (\alpha_i^\vee(u))$

$= [\alpha_i^\vee(u)] - [1]$.

$$\boxed{\begin{array}{l} \text{Ring structure on } \mathbb{Z}_{\mathbb{A}^1}[T]: \\ (u)_i (v)_j = (v)_j (u)_i \\ (u)_i (v)_i = (\eta (u)(v))_i \end{array}}$$

$$\bigotimes_{i=1}^r \overbrace{\mathbb{Z}_{\mathbb{A}^1}[\mathbb{G}_m]}^{\mathbb{Z} \oplus K_1^{MW}} \xrightarrow{\simeq} \mathbb{Z}_{\mathbb{A}^1}[T]$$

$$\| $$

$$\mathbb{Z} \oplus \Big(\bigoplus_{1 \le i_1 < \cdots < i_s \le r} K_s^{MW} \Big)$$

$$(u_1, \cdots, u_s) \longmapsto (u_1)_{i_1} \cdots (u_s)_{i_s} .$$

$C_*^{cell}(G)$ is a complex of right and left $\mathbb{Z}_{\mathbb{A}^1}[T]$-mod.

$$C_2^{cell}(G) \xrightarrow{\partial_2} C_1^{cell}(G) \xrightarrow{\partial_1} C_0^{cell}(G) \to 0$$
$$C_0^{cell}(G) = \mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega}_0 T]$$

$$C_2^{cell}(G) \to Z_1^{cell}(G) \to H_1^{cell}(G) \to 0$$

G simply connected $\Rightarrow$ $\mathbb{Z}_{\mathbb{A}^1}[T]$- action on $H_*^{cell}(G)$ is trivial.

$$C_2^{cell}(G) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \longrightarrow Z_1^{cell}(G) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \longrightarrow H_1^{cell}(G) \to 0.$$

Example: G: semisimple, split, rank 1.

$$C_*^{cell}(G) : \underbrace{\mathbb{Z}_{\mathbb{A}^1}(\mathbb{G}_m)}_{K_1^{MW}} \otimes \mathbb{Z}_{\mathbb{A}^1}[T] \longrightarrow \underbrace{\mathbb{Z}_{\mathbb{A}^1}[\dot{s}T]}_{\mathbb{Z} \oplus K_1^{MW}}$$

$W = \langle s \rangle$
$\updownarrow$
α

$$\mathbb{Z}_{\mathbb{A}^1}[\mathbb{G}_m \times T] \xrightarrow{\alpha^\vee \times id_T} \mathbb{Z}_{\mathbb{A}^1}[T \times T] \xrightarrow{\mu} \mathbb{Z}_{\mathbb{A}^1}[T] \xrightarrow{\simeq} \mathbb{Z}_{\mathbb{A}^1}[\dot{s}T]$$

1) $G = SL_2$, $\alpha^\vee : \mathbb{G}_m \xrightarrow{\sim} T$ $\quad id_{\mathbb{G}_m}$

$$C_*^{cell}(SL_2) : \quad K_2^{MW} \oplus K_1^{MW} \xrightarrow{\begin{pmatrix} \eta & 0 \\ id & 0 \end{pmatrix}} K_1^{MW} \oplus \mathbb{Z}$$

$$H_0^{cell}(SL_2) = \mathbb{Z} \qquad H_1^{cell}(SL_2) = K_2^{MW} .$$

2) $G = PGL_2$, $\alpha^\vee : \mathbb{G}_m \longrightarrow T$ square map

$$C_*^{cell}(PGL_2) : \quad K_2^{MW} \oplus K_1^{MW} \xrightarrow{\begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}} K_1^{MW} \oplus \mathbb{Z}$$

$$H_0^{cell}(PGL_2) = K_1^{MW}/h \oplus \mathbb{Z} = I \oplus \mathbb{Z} = K_0^{MW}$$
$$H_1^{cell}(PGL_2) = K_2^{MW} \oplus_h K_1^{MW} .$$

PGL_2 is <u>not</u> $\mathbb{A}^1$-connected $\quad \pi_0^{\mathbb{A}^1}(PGL_2) = K_1^M/2$

$$1 \to \underline{K}_2^{MW} \to \pi_1^{\mathbb{A}^1}(PGL_2) \to \mu_2 \to 1$$

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Differentials in $C_*^{cell}(G)$:

Fix a reduced expression

$$\omega_0 = \sigma_\ell \cdots \sigma_1 \quad , \quad \sigma_i \in \{s_1, \cdots, s_r\}$$

Set

$$\omega_i := \omega_0 s_i = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1$$ for a unique $\lambda_i \in \{1, \cdots, \ell\}$.

Write $\quad \omega_0 = \omega_i'' \sigma_{\lambda_i} \omega_i'$.

Set $\quad \tilde{\omega}_i := \dot{\omega}_i'' \dot{\omega}_i' \in N_G(T)$.

The differential

$$\partial_1 : C_1^{cell}(G) \to C_0^{cell}(G)$$

is a $\mathbb{Z}_{\mathbb{A}^1}[T]$-module homomorphism given by

$$\underset{\omega_0 s_i}{K_1^{MW} \otimes \mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega}_i T]} \longrightarrow \underset{\omega_0}{\mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega}_0 T]}$$

$$(u) \otimes [\tilde{\omega}_i \cdot 1] \longmapsto (\tilde{\omega}_0 \cdot \alpha_i^\vee(u))$$

$$\mathbb{Z}_{\mathbb{A}^1}(\tilde{\omega}_0 T) \subset \mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega}_0 T]$$

(uses $\tilde{\omega}_i'^{-1}(\beta_{\lambda_i}^\vee) = \alpha_i^\vee$)

root $\leftrightarrow \sigma_{\lambda_i}$ $\quad$ coroot $\leftrightarrow s_i$

Key Lemma :

$$\left(\bigoplus_i K_2^{MW}\right) \oplus \left(\bigoplus_{i<j} K_2^{M}\right) \xrightarrow{\sim} Z_1^{cell}(G) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

$$(u,v) \longmapsto \phi_i(u,v) = (u) \otimes (\tilde{\omega}_i \cdot \alpha_i^\vee(v)) - (\eta(u)(v)) \otimes [\tilde{\omega}_i \cdot 1]$$

$$(u,v) \mapsto \delta_{ij}(u,v) = (u) \otimes (\tilde{\omega}_i \cdot \alpha_j^\vee(v)) - (v) \otimes (\tilde{\omega}_j \cdot \alpha_i^\vee(u)) .$$

Degree 2 differential :

$$\omega_0 = \sigma_\ell \cdots \sigma_1$$

$$\omega_i = \omega_0 s_i = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \sigma_1$$

$$\omega_{[i,j]} = \omega_0 s_i s_j = \sigma_\ell \cdots \hat{\sigma}_{\lambda_i} \cdots \hat{\sigma}_{\mu_j} \cdots \sigma_1$$

for a unique $\mu_j \in \{1, \dots, \ell\} \setminus \{\lambda_i\}$.

Two cases : $\lambda_i > \mu_j$ or $\underbrace{\lambda_i < \mu_j}$

Reduce to $\lambda_i > \mu_j$ case

$W_{\{i,j\}} \subset W$

$u_0 v_0 = \omega_0$

u_0 longest in $W_{\{i,j\}}$

Key Proposition :

$C_2^{cell}(G) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z} \longrightarrow Z_1^{cell}(G) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$

is given by

$$\left(K_2^{MW} \otimes \mathbb{Z}_{\mathbb{A}^1}[\tilde{\omega}_{[i,j]} T]\right) \otimes_{\mathbb{Z}_{\mathbb{A}^1}[T]} \mathbb{Z}$$

—

$$(u,v) \otimes [\tilde{\omega}_{[i,j]} \cdot 1] \longmapsto \begin{cases} \overline{\delta}_{ji}(v,u) & \text{if } \alpha_i, \alpha_j \text{ not adjacent} \\ \overline{\delta}_{ji}(v,u) + (n_{ji})_\varepsilon \, \overline{\phi}_j(v,u) & \text{else, where} \end{cases}$$

$$n_{ji} = -\langle \alpha_j, \alpha_i^\vee \rangle$$

G : $\mathbb{A}^1$-connected $\Rightarrow$

$\pi_1^{\mathbb{A}^1}(G) = H_1^{\mathbb{A}^1}(G) = H_1^{\mathrm{cell}}(G)$.

Main theorem : k : perfect field.

Let G be as above. Fix a maximal k-split torus T and a Borel B in $G \supset T$. Let α be a long root in the Dynkin diagram of G ; $S_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$. The inclusion $S_\alpha \hookrightarrow G$ induces an epimorphism

$$\pi_1^{\mathbb{A}^1}(S_\alpha) \twoheadrightarrow \pi_1^{\mathbb{A}^1}(G).$$

(a) This induces a canonical isomorphism

$$\underline{K}_2^M \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(G)$$

if G is not of symplectic type.

$$\left(\pi_1^{\mathbb{A}^1}(SL_2) = \underline{K}_2^{MW} \longrightarrow \underline{K}_2^{MW}/\eta = \underline{K}_2^M . \right)$$

(b) This induces a non-canonical isomorphism

$$\underline{K}_2^{MW} \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(G)$$

if G is of symplectic type.