Lecture 3

Kronecker-type identities with applications to classical results on sums of squares and sums of triangular numbers.

Last time.

- mock theta functions have many forms
- changing between forms leads to a better understanding of mock theta functions.
- we introduced a <u>heuristic</u> relating Appell-Lerch functions to divergent partial theta functions

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1}xz}$$

$$j(z;q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k$$

$$m(x,q,z) \sim \sum_{r\geq 0} (-1)^r x^r q^{-\binom{r+1}{2}} \qquad (*)$$

$0<|q|<1$

the heuristic led to

- new identities for the $m(x,q,z)$ function
- formulas that expand Hecke-type double-sums in terms of $m(x,q,z)$ functions
- new proofs of the six identities for Ramanujan's four tenth order mock theta functions,

where the four functions and the six identities where all found in Ramanujan's Lost Notebook, today we will demonstrate how the heuristic leads to a new identity which gives new proofs of classical results in number theory such as

1) Legendre - Gauss local to global principle

$r_3(n) > 0 \iff n \neq 4^a(8b+7)$

2) Gauss's Eureka Theorem

$\text{num} = \Delta + \Delta + \Delta$

$\Delta = k(k+1)/2, \quad k = 0, 1, 2, \ldots$

$r_3(n) = \#$ ways to write n as a sum of three $\square$'s

Review

Many forms of mock theta functions

1) q-hypergeometric

$$f_0(q) := \sum_{n=0}^{\infty} q^{n^2} \Big/ (-q;q)_n$$

$$= 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \cdots$$

2) Hecke-type double-sum

$$f_0(q) \prod_{n=1}^{\infty} (1-q^n)$$

$$= \sum_{n=0}^{\infty} q^{5n^2/2 + n/2} (1-q^{4n+2}) \sum_{j=-n}^{n} (-1)^j q^{-j^2}$$

Review

Many forms of mock theta functions

3) Appell-Lerch functions

$$f_0(q) = m(q^{14}, q^{30}, q^{14}) + m(q^{14}, q^{30}, q^{29})$$
$$+ q^{-2} m(q^{4}, q^{30}, q^{4}) + q^{-2} m(q^{4}, q^{30}, q^{14})$$

where

$$m(x, q, z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1 - q^{n-1} x z}$$

$$j(z;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n$$

4) Fourier Coefficients of meromorphic Jacobi forms (skip)

Review:

we introduced a heuristic

$$m(q_\lambda x, q_\lambda, z) = 1 - x\,m(x, q_\lambda, z)$$

$$* \quad m(x, q_\lambda, z) = 1 - q_\lambda^{-1} x\, m(q_\lambda^{-1} x, q_\lambda, z)$$

iterate

$$m(x, q_\lambda, z) = 1 - q_\lambda^{-1} x\, m(q_\lambda^{-1} x, q_\lambda, z)$$

$$= 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2\, m(q_\lambda^{-2} x, q_\lambda, z)$$

$$= 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2 - q_\lambda^{-6} x^3\, m(q_\lambda^{-3} x, q_\lambda, z)$$

$$\sim 1 - q_\lambda^{-1} x + q_\lambda^{-3} x^2 - q_\lambda^{-6} x^3 + q_\lambda^{-10} x^4 + \sim$$

$$m(x, q_\lambda, z) \sim \sum_{r \geq 0} (-1)^r x^r q_\lambda^{-\binom{r+1}{2}}$$

$*$

Review

let us break $(*)$ into two parts depending on parity of r

$$m(x,q_\lambda,z) \sim \sum_{r\geq 0} (-1)^r q_\lambda^{-\binom{r+1}{2}} x^r \quad (*)$$

$$\sim \sum_{r\geq 0} q_\lambda^{-\binom{2r+1}{2}} x^{2r} - \sum_{r\geq 0} q_\lambda^{-\binom{2r+2}{2}} x^{2r+1}$$

$$\sim \sum_{r=0}^{\infty} (-1)^r q_\lambda^{-4\binom{r+1}{2}} (-q_\lambda x^2)^r$$

$$-q_\lambda^{-1} x \sum_{r=0}^{\infty} (-1)^r q_\lambda^{-4\binom{r+1}{2}} (-q_\lambda^{-1} x^2)^r$$

$$\sim m(-q_\lambda x^2, q_\lambda^4, z_0) - q_\lambda^{-1} x\, m(-q_\lambda^{-1} x^3, q_\lambda^4, z_1)$$

Review

Notation

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1}xz}$$

$$j(z;q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k$$

$$J_{a,m} := j(q^a; q^m)$$

$$\overline{J}_{a,m} := j(-q^a; q^m)$$

$$J_m := J_{m,3m} = j(q^m; q^{3m})$$

false theta fn

partial theta fn

Review

$$m(x,q,z) = m(-qx^2, q^4, z') - q^{-1}x\, m(-q^{-1}x^2, q^4, z')$$

$$+ \frac{z' \overline{J}_2^{\,3}}{j(xz;q)\, j(-qx^2z;q^2)} \left[\frac{j(-qx^2zz';q^2)\, j(z^2/z';q^4)}{j(z';q^4)\, j(z;q^2)} - xz \frac{j(-q^2x^2zz';q^2)\, j(q^2z^2/z';q^4)}{j(z';q^4)\, j(qz;q^2)} \right]$$

more general formulas exist!

Review:

$$m(x,q,z) = \sum_{r=0}^{n-1} q^{-\binom{r+1}{2}} (-x)^r \cdot m\left(-q^{\binom{n}{2}-nr}(-x)^n, q^{n^2}, z'\right)$$

$$+ \frac{z' J_n^3}{j(xz;q)\, j(z';q^{n^2})} \cdot \sum_{r=0}^{n-1} \left[q^{\binom{r}{2}} (-xz)^r \cdot j\left(-q^{\binom{n}{2}+r}(-x)^n z z'; q^n\right) \cdot \frac{j\left(q^{nr} z^n / z'; q^{n^2}\right)}{j\left(-q^{\binom{n}{2}}(-x)^n z'; q^n\right) j\left(q^r z; q^n\right)} \right]$$

Review

Hecke-type double-sums

building block

$$f_{a,b,c}(x,y,q) := \left(\sum_{r,s \geq 0} - \sum_{r,s < 0} \right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}}$$

Converges absolutely $a,b,c > 0$

weight system depends on region being summed over

Review

$f_{1,2,1}(x,y,q)$

$$= j(y;q)\cdot m(q^2x/y^2, q^3, -1)$$

$$+ j(x;q)\, m(q^2y/x^2, q^3, -1)$$

$$- y\,\frac{1}{j(-1;q^3)}\,\frac{\bar{J}_3^3\, j(-x/y;q)\, j(q^2xy;q^3)}{j(-qy^2/x;q^3)\, j(-qx^2/y;q^3)}$$

$b^2 - ac > 0$

Examples

$f_{1,2,1}(q,q,q)$

$=j(q;q)\,m(q,q^3,-1)$

$\quad +j(q;q)\,m(q,q^3,-1) + \bar{J}_1^2$

$j(q;q)=0$

so

$$f_{1,2,1}(q,q,q) = \bar{J}_1^2 = \prod_{i=1}^{\infty}(1-q^i)$$

Recall $j(x;q)=0 \iff x=q^n \quad n\in\mathbb{Z}$

$m(x,q,z)$ singularities from $\rightarrow$

$z = q^n \quad n\in\mathbb{Z}$

$xz = q^m \quad m\in\mathbb{Z}$

Examples $\rightarrow$ 6th order mock theta fn

$$\bar{J}_{1,4} \cdot \phi(q) = f_{1,2,1}(q,-q,q)$$

Bailey Pair Technology

$$f_{1,2,1}(q,-q,q)$$

$$= j(-q;q)m(q,q^3,-1)$$

$$+j(q;q)m(-q,q^3,-1) + 0$$

$$= j(-q;q)\,m(q,q^3,-1)$$

or $\phi(q) = 2m(q,q^3,-1)$

use:

$j(-q;q) = 2\,\bar{J}_{1,4}$ product rearrangement

Review

Notation

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{1-q^{n-1}xz}$$

$$j(z;q) := \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k$$

$$J_{a,m} := j(q^a; q^m)$$

$$\overline{J}_{a,m} := j(-q^a; q^m)$$

$$J_m := J_{m,3m} = j(q^m; q^{3m})$$

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1} (1-q^i x)$$

Application Review

The four tenth order mock thetas.

$$\phi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}} \qquad \psi(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}}}{(q;q^2)_{n+1}}$$

$$X(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \qquad \chi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}$$

One of the six identities reads

$$\phi(q) - q^{-1}\psi(-q^4) + q^{-2}\chi(q^8)$$

$$= \frac{j(-q;q^2)\, j(-q^2;-q^{10})}{j(q^2;q^8)}$$

Celebrated results of

	two ids	Inv Math '99
Choi	two ids	Adv Math '00
	two ids	Proc LMS '07

Application Review

q-hypergeometric   LHS

↓ Bailey's Lemma

Hecke-type double-sum

↓ Hecke-type to $m(x,q,z)$ formula

$m(x,q,z)$ form

↓ new $m(x,q,z)$ properties

theta functions   RHS

Today

- we will review classical number theory results on the representations of a number n as a sum of k squares or as a sum of k triangular numbers

$\square = k^2$ $\quad\Delta = k(k+1)/2$

- demonstrate how a q-series identity known to Kronecker gives as special cases
  - Lagrange's four-square thm
  - Fermat's two-square thm

Today

- use the heuristic as guide to find a double-sum analog of Kronecker's identity.

special cases of the new double-sum analog yield

- Legendre-Gauss local-to-global principle

  $r_3(n) > 0 \iff n \neq 4^a (8b+7)$

- Gauss's Eureka theorem

  $\text{num} = \Delta + \Delta + \Delta$

  $\Delta = k(k+1)/2$

$r_3(n) =$ # ways to write n as a sum of 3 squares

Today

More detailed plan

- Give examples of classical number theory results
  $\sum \square$'s $\qquad \sum \Delta$'s
- Recall Kronecker's identity
  - Prove Kronecker's identity using Ramanujan's ${}_1\psi_1$ summation
  - Prove Ramanujan's ${}_1\psi_1$ summation
  - demonstrate that special cases yield
    - Lagrange's four-square thm
    - Fermat's two-square thm

Today

More detailed plan

- use heuristic to find double-sum analog of Kronecker
- sketch proof
- demonstrate that special cases yield
  - $r_3(n) > 0 \iff n \neq 4^a(8b+7)$
  - $n = \Delta + \Delta + \Delta$, $n = k(k+1)/2$

Compute examples

$$r_3(n) > 0 \iff n \neq 4^a(8b+7)$$

$n = 3 = \quad 1^2 + 1^2 + 1^2$

$$3 \neq 4^a(8b+7)$$

$a = 0, \quad b = 0$

$$4^a(8b+7) = 7$$

7 cannot be written as a sum of 3 $\square$'s

$3 = 1^2 + 1^2 + 1^2$ $\qquad$ $1 = 1^2 + 0^2 + 0^2$

$4 = 2^2 + 0^2 + 0^2$ $\qquad$ $2 = 1^2 + 1^2 + 0^2$

$5 = 4^2 + 1^2 + 0^2$

$6 = 4^2 + 1^2 + 1^2$

7 none.

Classical problems in number theory

- count the number of representations of a positive integer n
  - as a sum of k squares
  - as a sum of k triangular numbers

- Gauss's two-square theorem

$$\square_2(n) = 4 \sum_{d \mid n,\ d \text{ odd}} (-1)^{\frac{1}{2}(d-1)}$$

Jacobi's four & eight square theorem

$$\square_4(n) = 8 \sum_{d \mid n,\ 4 \nmid d} d$$

$$\square_8(n) = 16 \sum_{d \mid n} (-1)^{n+d} d^3$$

$\square_{2k} \sim$ elliptic functions

Compute Examples

—o—

$\square_2(n)$

$$n=2 \qquad \left.\begin{aligned} 2 &= 1^2+1^2 \\ &= (-1)^2+1^2 \\ &= (-1)^2+(-1)^2 \\ &= 1^2+(-1)^2 \end{aligned}\right\} 4$$

$$\square_2(n) = 4 \cdot \sum_{d \mid 4,\ d\ \text{odd}} (-1)^{\frac{1}{2}(d-1)}$$

$$= 4 \cdot (-1)^{\frac{1}{2}\cdot 0} \qquad\qquad d=1$$

$$= 4 \cdot 1 = 4$$

Legendre's formulas for sums of triangular numbers

$\Delta = k(k+1)/2$

$$\Delta_2(n) = \sum_{d \mid 4n+1} (-1)^{\frac{1}{2}(d-1)}$$

$$\Delta_4(n) = \sum_{d \mid 2n+1} d. \qquad \Delta_8(n) = \cdots$$

Kac & Wakimoto used denominator formulas for affine superalgebras to derive many inf. families of identities

conjectures - proofs by Zagier, Milne subsequent work

- Rosengren, Ono, Getz, Mahlberg.

Example: Kac & Wakimoto conjectured

$$\Delta_{4m^2}(n) = \frac{1}{4^{m(m-1)} \prod_{j=1}^{2m-1} j!} \cdot$$

$$\cdot \sum_{\substack{k_1 l_1 + k_2 l_2 + \dots + k_m l_m = 2n + m^2 \\ k_1 > k_2 > \dots > k_m \\ k_i \text{ positive},\ l_i \text{ odd positive}}} \prod_{i=1}^{m} k_i \prod_{1 \le i < j \le m} (k_i^2 - k_j^2)^2$$

Case $m=1$ gives Legendre's formula for $\Delta_4(n)$!

Compute Examples

—o—

$\Delta = k(k+1)/2 \qquad k = 0, 1, 2, 3, \ldots$

$\Delta = 0, 1, 3, 6, 10, \ldots$

$$\Delta_4(2) = \sum_{d \mid 5} d = 1 + 5 = 6$$

$$\left.\begin{aligned} 2 &= 1 + 1 + 0 + 0 \\ & 1 + 0 + 1 + 0 \\ & 1 + 0 + 0 + 1 \\ & 0 + 1 + 1 + 0 \\ & 0 + 1 + 0 + 1 \\ & 0 + 0 + 1 + 1 \end{aligned}\right\} 6$$

here we will put classical number theory results into the setting of Kronecker-type identities

Questions

- can they be further generalized?

- if so, what kinds of new identities are there?

- are there more general building blocks?

Reminder:

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1} (1-q^i x)$$

$$(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty} (1-q^i x)$$

$$j(x;q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n$$

$$\text{JTPid} \quad = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty$$

$$j(x_1, x_2, \ldots, x_n;q)$$

$$:= j(x_1;q) j(x_2;q) \cdots j(x_n;q)$$

$$J_{a,m} := j(q^a;q^m) \quad \overline{J}_{a,m} := j(-q^a;q^m)$$

$$J_m = J_{m,3m} = \prod_{i=1}^{\infty} (1-q^{mi})$$

The following identity was known to Kronecker

For $x, y \in \mathbb{C} \setminus \{0\}$, $0 < |q| < |x| < 1$ and y neither zero nor an integral power of q

$$\sum_{r \in \mathbb{Z}} \frac{x^r}{1 - q^r y} = \frac{(q)_\infty^2 \, (xy, q/xy; q)_\infty}{(x, q/x, y, q/y; q)_\infty}$$

or in theta function notation

$$= \frac{J_1^3 \, j(xy; q)_\infty}{j(x; q) \, j(y; q)}$$

Alternate from of Kronecker's id.

For $x, y \in \mathbb{C}\setminus\{0\}$, $0<|q|<|x|<1$ &

$0<|q|<|y|<1$ then we have the

more symmetric form

$$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) q^{rs} x^r y^s$$

$$= \frac{(q)_\infty^2 (xy, q/xy; q)_\infty}{(x, q/x, y, q/y; q)_\infty}$$

or in theta function notation

$$= \frac{J_1^3 \, j(xy;q)}{j(x;q) j(y;q)}$$

- how do we prove Kronecker's identity?
- what are its applications to classical no. theory results?

Proof of Kronecker's identity

- state and prove Ramanujan's ${}_1\psi_1$ summation
- Use Ramanujan's ${}_1\psi_1$ summation to prove Kronecker's identity.

Survey article by Warnaar on ${}_1\psi_1$ summation

Ramanujan's ${}_1\psi_1$ summation

& Proof

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useful facts

$$(a;q)_n = \prod_{i=0}^{n-1} (1-q^i a)$$

$$(a;q)_{-n} = (-q/a)^n q^{\binom{n}{2}} \Big/ (q/a;q)_n$$

q- binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}$$

Ramanujan's ${}_1\psi_1$ summation & Proof

$${}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}; q, z\right) := \sum_{r \in \mathbb{Z}} \frac{(a)_r}{(b)_r} z^r$$

$$= \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty} \qquad (*)$$

$|b/a| < |z| < 1$

it is understood that $a, b \neq 0$

and that $a, q/b \in \{q, q^2, \ldots\}$

- Ramanujan did not provide a proof
- A short proof was discovered by M. Ismail 1977 Proc AMS

2 pages

Proof of Ramanujan's ${}_1\psi_1$ - Ismail

$${}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q,z\right) := \sum_{n=-\infty}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n} + \sum_{n=-\infty}^{-1} \frac{(a;q)_n z^n}{(b;q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(b;q)_n} + \sum_{n=1}^{\infty} \frac{(qb^{-1};q)_n}{(qa^{-1};q)_n} \left(\frac{b}{az}\right)^n$$

So ${}_1\psi_1$ is an analytic fn of b provided $|q|<1$, $|z|<1$, $|b|<|az|$

(*) reduces to the q-binomial thm whenever $b=q^m$ $m\in\mathbb{Z}$

(*) is valid in general since it holds on a convergent sequence within the domain of analyticity.

Proof of Kronecker's identity

$(b, z) \mapsto (aq, b)$ in ${}_1\psi_1$

$$\sum_{r\in\mathbb{Z}} \frac{(a)_r z^r}{(b)_r} = \frac{(az, q/az, b/a, q; q)_\infty}{(z, b/az, q/a, b; q)_\infty}$$

$$\text{LHS} \to \sum_{r\in\mathbb{Z}} \frac{(a)_r b^r}{(aq)_r} = (1-a)\sum_{r\in\mathbb{Z}} \frac{b^r}{(1-aq^r)}$$

$$\text{RHS} \to \frac{(ab, q/ab, q, q; q)_\infty}{(b, q/b, q/a, aq; q)_\infty}$$

divide both sides by $(1-a)$

and we are done

Applications of Kronecker's identity

$$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) q^{rs} a^r b^s$$

$$= \frac{(q)_\infty^2 \, (ab, q/ab; q)_\infty}{(a, q/a, b, q/b; q)_\infty}$$

1) $a, b \to -1$

i) Jacobi's 4-square thm

$$r_4(n) = 8 \sum_{d\geq 1,\, 4\nmid d,\, d|n} d$$

ii) Lagrange's 4-square thm

2) $a, b^2 \to -1$

i) Gauss & Lagrange

$$r_2(n) = 4\left(d_1(n) - d_3(n)\right)$$

ii) Fermat's two-square thm

$r_k(n)$ = # reps of n as sum of k $\square$'s

$d_k(n)$ = # divisors of n of form $4m+k$

Proof of Jacobi's four square thm

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Generating function for $r_s(n)$

\# ways to write n as sum of s $\square$'s

$$R_s(q) := \sum_{n \geq 0} r_s(n) (-q)^n$$

Claim: $R_s(q) := \left(\sum_{m \in \mathbb{Z}} (-1)^m q^{m^2} \right)^s$

$$= \left(\frac{(q;q)_\infty}{(-q;q)_\infty} \right)^s$$

in particular

$$\sum_{n=0}^{\infty} r_4(n) (-q)^n = \left(\frac{(q;q)_\infty}{(-q;q)_\infty} \right)^4$$

Proof of Jacobi's four sq. thm

idea

make LHS of Kronecker look like

$$8 \sum_{\substack{d \geq 1 \\ d \mid m,\ 4 \nmid d}} d$$

make RHS of Kronecker look like

$$\left(\frac{(q;q)_\infty}{(-q;q)_\infty} \right)^4$$

$$= \sum_{n=0}^{\infty} r_4(n) (-q)^n$$

Proof of Jacobi's four square thm

Proof of Claim

$$R_s(q) = \left(\sum_{m \in \mathbb{Z}} (-1)^m q^{m^2} \right)^s$$

$$= \left(\sum_{m \in \mathbb{Z}} (-1)^m q^m q^{2\binom{m}{2}} \right)^s$$

$$= \left(j(q; q^2) \right)^s \quad \leftarrow \text{def}$$

JTP id

$$= \left((q;q^2)_\infty^2 \, (q^2;q^2)_\infty \right)^s$$

$$= \left((q;q)_\infty \, (q;q^2)_\infty \right)$$

$$= \left((q;q)_\infty^2 \Big/ (q^2;q^2)_\infty \right)^s$$

Proof of Jacobi's four square thm

$$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) q^{rs} a^r b^s$$

$$= \frac{(q)_\infty^2 \,(ab, q/ab; q)_\infty}{(a, q/a, b, q/b; q)_\infty}$$

let $a, b \to -1$

but first we have to get rid of singularities

Proof of Jacobi's four sq. thm

We rewrite LHS

$$\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) q^{rs} a^r b^s$$

$$\sum_{r,s\geq 0} q^{rs} a^r b^s$$

$$= \left(\sum_{\substack{r=0\\ s>0}} + \sum_{\substack{r>0\\ s=0}} + \sum_{\substack{r=0\\ s=0}} + \sum_{\substack{r>0\\ s>0}}\right) q^{rs} a^r b^s$$

$$= \frac{1}{1-b} + \frac{1}{1-a} - 1 + \sum_{r,s>0} q^{rs} a^r b^s$$

LHS is now

$$\frac{1}{1-b} + \frac{1}{1-a} - 1 + \sum_{r,s=1}^{\infty} q^{rs}\left(a^r b^s - a^{-r} b^{-s}\right)$$

Proof of Jacobi's four sq thm

multiple both LHS & RHS

by $(1-a)(1-b)/(1-ab)$

$$1+\frac{(1-a)(1-b)}{1-ab}\sum_{r,s=1}^{\infty} q^{rs}\left(a^r b^s - a^{-r} b^{-s}\right)$$

$$= \frac{(q)_\infty^2\,(abq, q/ab; q)_\infty}{(qa, q/a, qb, q/b; q)_\infty}$$

Let $a=b$ in LHS

$$1+\frac{(1-a)^2}{(1-a^2)}\sum_{r,s=1}^{\infty} q^{rs}\left(a^{r+s} - a^{-r-s}\right)$$

$$= 1+\frac{(1-a)^2}{(1-a^2)}\sum_{r,s=1}^{\infty} q^{rs} a^{-r-s}\left(a^{2(r+s)} - 1\right)$$

$$= 1+(1-a)^2\sum_{r,s=1}^{\infty} q^{rs} a^{-r-s} \cdot \left(1+a^2+a^4+\cdots+a^{2(r+s-1)}\right)$$

Proof of Jacobi's four sq thm

now let $q = -1$

$$\text{LHS} = 1 - 4\sum_{r,s=1}^{\infty} q^{rs} (-1)^{r+s} (r+s)$$

$$= 1 - 4\sum_{r,s=1}^{\infty} q^{rs} (-1)^{r+s} r - 4\sum_{r,s=1}^{\infty} q^{rs} (-1)^{r+s} s$$

$$= 1 - 8\sum_{r,s=1}^{\infty} q^{rs} r (-1)^{r+s}$$

$$= 1 - 8\sum_{m=1}^{\infty} q^{m} \sum_{\substack{n,k=1 \\ nk=m}}^{\infty} n (-1)^{n+k}$$

$$= 1 + 8\sum_{m=1}^{\infty} (-q)^{m} \sum_{\substack{n,k=1 \\ nk=m}}^{\infty} n (-1)^{(n+1)(k+1)}$$

$$= 1 + 8\sum_{m=1}^{\infty} (-q)^{m} \sum_{\substack{d \geq 1 \\ 4 \nmid d,\ d \mid m}} d.$$

Proof of Jacobi's four sq. thm.

LHS $\quad a=b,\ a \to -1$

$$= 1 + 8\sum_{m=1}^{\infty} (-q)^m \sum_{\substack{d\geq 1 \\ 4\nmid d,\ d\mid m}} d$$

RHS $\quad a=b,\ a \to -1$

$$\frac{(q)_\infty^2\,(q)_\infty^2}{(-q)_\infty^4} = \left((q)_\infty \big/ (-q)_\infty \right)^4$$

so

$$1 + 8\sum_{m=1}^{\infty} (-q)^m \sum_{\substack{d\geq 1 \\ 4\nmid d,\ d\mid m}} d = \left((q)_\infty \big/ (-q)_\infty \right)^4$$

$$= \sum_{m=0}^{\infty} r_4(m)\,(-q)^m$$

$$r_4(m) = 8 \sum_{\substack{d\geq 1 \\ 4\nmid d,\ d\mid m}} d.$$

Double-sum analog of Kronecker's id.

Q:

what is

$$\left(\sum_{s,t\geq 0} - \sum_{s,t<0}\right) \frac{q^{st}y^s z^t}{1-xq^{s+t}}$$

or in Hecke-type form

$$\left(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\right) q^{rs+rt+st} x^r y^s z^t$$

Idea: set up functional eqn

iterate functional eqn

use heuristic as guide

numerical work to find theta fn.

Recall

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k}{1 - q^{k-1} x z}$$

$$j(z;q) = \sum_{k=-\infty}^{\infty} (-1)^k z^k q^{\binom{k}{2}}$$

$$m(x,q,z) \sim \sum_{r \geq 0} (-1)^r q^{-\binom{r+1}{2}} x^r$$

The functional equation

$F(x,y,z;q)$

$$:= \left(\sum_{r,s,t \geq 0} + \sum_{r,s,t<0} \right) q^{rs+rt+st} x^r y^s z^t$$

$$F(x,y,z;q) = x F(x, qy, qz; q) + \bar{J}_1^3 j(yz;q) \big/ j(y;q) j(z;q)$$

Proof - straight forward shift of indices

- Kronecker's id.

Iterate the functional equation

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q)$$

$n=1$ $\quad j(qx;q) = -x^{-1} j(x;q)$

$$F(x,y,z;q) = \frac{J_1^3\, j(yz;q)}{j(y;q)\,j(z;q)} + xF(x,qy,qz;q)$$

apply functional eqn to

$$xF(x,qy,qz;q)$$

$$= x\frac{J_1^3\, j(q^2yz;q)}{j(qy;q)\,j(qz;q)} + x^2 F(x,q^2y,q^2z;q)$$

$$= \frac{x}{qyz}\,\frac{J_1^3\, j(yz;q)}{j(y;q)\,j(z;q)} + x^2 F(x,q^2y,q^2z;q)$$

So

$$F(x,y,z,q)$$

$$= \frac{\bar{J}_1^3 j(yz;q)}{j(y;q)j(z;q)} + x F(x,qy,qz,q)$$

$$= \frac{\bar{J}_1^3 j(yz;q)}{j(y;q)j(z;q)}\left(1+\frac{x}{qyz}\right) + x^2 F(x,q^2y,q^2z,q)$$

$= \ldots$ iterate

$$\sim \frac{\bar{J}_1^3 j(yz;q)}{j(y;q)j(z;q)}\left(1+\frac{x}{qyz}+\frac{x^2}{q^{-4}y^2z^2}+\cdots\right.$$

$$\sim \frac{\bar{J}_1^3 j(yz;q)}{j(y;q)j(z;q)}\sum_{k\geq 0}(-1)^k\left(\frac{-qx}{yz}\right)^k q^{-2\binom{k+1}{2}}$$

iterate functional eq^n

use heuristic

$$m(x,q,z) \sim \sum_{r=0}^{\infty} (-1)^r x^{-r} q^{-\binom{r+1}{2}}$$

$$F(x,y,z;q) \sim \frac{J_1^3\, j(yz;q)}{j(y;q)\, j(z;q)}\, m\left(-\frac{qx^2}{yz}, q, *\right)$$

Symmetry suggests

$$F(x,y,z;q) = \frac{J_1^3\, j(yz;q)}{j(y;q)\, j(z;q)}\, m\left(-\frac{qx^2}{yz}, q, *\right)$$

$+\,\text{idem}(x;y,z) + \text{theta}$

↑ means previous term repeated once w/ y & x swapped, once w/ x, z swapped.

Final Formula

$$F(x,y,z;q) = \frac{\bar{J}_{1,1}^{3}(yz;q)}{j(y;q)j(z;q)} m\left(\frac{-qx}{yz}, q^2, qyz\right)$$

$+$ idem $(x;y,z)$

$$-2\cdot\frac{\bar{J}_1^3\bar{J}_2^3}{j(x,y,z;q)}\cdot\frac{j(xy,xz,yz;q^2)}{j(-x,-y,-z;q^2)}$$

Applications

I) $x=y=z \longrightarrow -1$

Legendre Gauss

$r_3(n)>0 \iff n \neq 4^a(8b+7)$

II) $q \longmapsto q^2 \quad x=y=z \to q$

Gauss' Eureka Thm

num $= \Delta+\Delta+\Delta$, $\Delta = k(k+1)/2$

Last items:

- Sketch proof of analog
- Sketch proof of Gauss's Eureka Theorem

from last time:

type of proof seen before

$$m(x,q,z_1) - m(x,q,z_0)$$

$$= \frac{z_0 J_1^3 j(z_1/z_0;q) j(x z_0 z_1;q)}{j(z_0;q) j(z_1;q) j(x z_0;q) j(x z_1;q)}$$

$F(x, z_1, z_0, q) = \text{LHS}$

$G(x, z_1, z_0, q) = \text{RHS}$

$H = F - G$

$H(x) = \sum_{m \in \mathbb{Z}} c_m x^m$ Laurent Series $x \neq 0$

$H(qx) = -x H(x)$

$$H(x) = c_0 \sum_{m \in \mathbb{Z}} (-1)^m q^{-\binom{m+1}{2}} x^m$$

Ratio test $\Rightarrow c_0 = 0$

Proof of double-sum analog

$$F(x,y,z;q) := \left(\sum_{s,t\geq 0} - \sum_{s,t<0}\right) \frac{q^{st} y^s z^t}{1-xq^{s+t}}$$

$$G(x,y,z;q) := \frac{J_1^3 j(yz;q)}{j(y;q)j(z;q)} m\left(-\frac{qx}{yz}, q^2, qyz\right)$$

$$+ \text{idem}\,(x; y, z)$$

$$-2\frac{J_1^3 J_2^3}{j(x,y,z;q)} \cdot \frac{j(xy, xz, yz; q^2)}{j(-x,-y,-z;q^2)}$$

Proof of double-sum analog

Define

$H(x,y,z;q) = F(x,y,z;q) - G(x,y,z;q)$

Fix y, z, q

- We want to show $H(x)$ is analytic for all $x \neq 0$
- Show singularities are at most simple poles
- Show residues at poles sum to zero
- Show $H(x)$ satisfies the functional eqn.

$$H(q^2 x, y, z; q) = \frac{xq}{yz} H(x,y,z;q)$$

Proof of double-sum analog

$H(x,y,z;q) = F(x,y,z;q) - G(x,y,z;q)$

is analytic for all $x \neq 0$ so it can be written as a Laurant series in x valid for all $x \neq 0$

$$** \quad H(x,y,z;q) = \sum_{m \in \mathbb{Z}} C_m x^m$$

C_m may depend on y, z, q

Insert $(*)$ $H(q^2 x) = \frac{xq}{yz} H(x)$

into $(**)$ to have

$$\sum_{m \in \mathbb{Z}} C_m (q^2 x)^m = \frac{xq}{yz} \sum_{m \in \mathbb{Z}} C_m x^m$$

Proof of double-sum analog

So $c_{m+1} = \frac{q_\lambda^{-1-2m}}{yz} \cdot c_m$

in general $c_k = y^{-k} z^{-k} q_\lambda^{-k^2} c_0$

Hence $H(x, y, z; q_\lambda)$

$$= c_0 \sum_{k \in \mathbb{Z}} y^{-k} z^{-k} q_\lambda^{-k^2} x^k$$

Because $H(x)$ is analytic for all $x \neq 0$ we can use the ratio test to conclude that $c_0 = 0$

Proof of double-sum analog

ratio test.

assume $c_0 \neq 0$

look at ratio of consecutive terms

as $k \to \infty$

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{y^{k+1} z^{k+1} q^{(k+1)^2}_{\chi}} \cdot \frac{y^k z^k q^{k^2}_{\chi}}{x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x}{yz} \cdot \frac{1}{q^{2k+1}_{\chi}} \right| \to \infty$$

all $x \neq 0$!

Gauss's Eureka Thm.

Gauss discovered that every positive integer is a sum of three triangular numbers

EYPHKA! num $= \Delta + \Delta + \Delta$

$\Delta = k(k+1)/2 \quad k = 0, 1, 2, \ldots$

$\Delta = 0, 1, 3, 6, 10, 15, \ldots$

- does not imply that the triangular numbers are different

  $20 = 10 + 10 + 0$

- does not imply that a solution with exactly three nonzero triangular numbers must exist

  $1 = 1 + 0 + 0$

Proof of Eureka Thm

—o—

Preliminaries

$$j(z;q) = \sum_{k\in\mathbb{Z}} (-1)^k q^{\binom{k}{2}} z^k$$

$$\text{JTPId} \quad = (z)_\infty (q/z)_\infty (q)_\infty$$

Classical partial fraction decomposition for Jacobi's theta fn

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\binom{n+1}{2}}}{1-q^n z} = \frac{(q;q)_\infty^2}{(z, q/z;q)_\infty}$$

$r_{3\Delta}(n) = $ # of ways to write n as a sum of three Δ's.

$$\sum_{n=0}^{\infty} r_{3\Delta}(n) q^n = \left(\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\right)^3$$

Proof of Eureka Theorem

claim
$$\sum_{n=0}^{\infty} r_{3\underline{a}}(n) q^n := \left(\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} \right)^3$$
$$\longrightarrow \quad = \left((q^2;q^2)^2_\infty \Big/ (q;q)_\infty \right)^3$$

Proof
$$\sum_{n=0}^{\infty} q^{\binom{n-1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} + \frac{1}{2}\sum_{n=0}^{\infty} q^{\binom{n+1}{2}}$$

$n \to -1-n$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}}$$
$$= \frac{1}{2}\sum_{n=\infty}^{\infty} (-1)^n (-q)^n q^{\binom{n}{2}}$$

Proof of Eureka Theorem

Proof of Claim

$$\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{1}{2}\, j(-q;q)$$

$$= \frac{1}{2} (-q;q)_\infty \left(\frac{q}{-q};q\right)_\infty (q;q)_\infty$$

recal $(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\cdots$

so $(-1;q) = (1-(-1))(1+q)(1+q^2)\cdots$

$$= 2\cdot(-q;q)_\infty$$

$$= (-q;q)_\infty^2 (q;q)_\infty$$

$$= (-q;q)_\infty (q^2;q^2)_\infty$$

$$= (q^2;q^2)_\infty^2 \Big/ (q;q)_\infty$$

Proof of Eureka Theorem

For $x, y, z \in \mathbb{C}\setminus\{0\}$

$0 < |q| < |x|, |y|, |z| < 1$

$$\left(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\right) q^{rs+rt+st} x^r y^s z^t$$

$$= \frac{(xy, q^2/xy; q^2)_\infty}{(x, y, z, q/x, q/y, q/z; q)_\infty} \cdot \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$$

$$\cdot \sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{k^2} (xy)^k}{1+q^{2k} z} + \text{idem}(z; x, y)$$

$$-2 \frac{(q^2;q^2)_\infty^2}{(x, y, z, q/x, q/y, q/z; q)_\infty} \cdot$$

$$\cdot \frac{(xy, xz, yz, q^2/xy, q^2/xz, q^2/yz; q^2)_\infty}{(-x, -y, -z, -q^2/x, -q^2/y, -q^2/z; q^2)_\infty}$$

Proof of Eureka Theorem

— o —

Idea

RHS look like

$$\sum_{n=0}^{\infty} r_{3\Delta}(n) = \left((q^2;q^2)_\infty^2 / (q)_\infty \right)^3$$

LHS

something that shows

$r_{3y}(n) \geq 1$ all n

Proof of Eureka

we set $q \mapsto q^2$ $x = y = z$, $x \to q$

The RHS has four pieces

the theta quotient $\to -2 \dfrac{(q^2;q^2)_\infty^2}{(q;q)_\infty^3}$

The three terms

with Appell-Lerch fns

each go to $(q^2;q^2)_\infty^6 \Big/ (q;q)_\infty^3$

where we used the partial fraction expansion of Jacobi's theta product

Summing the four pieces on the RHS

$$RHS = \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3} = \sum_{n=0}^{\infty} r_{3\Delta}(n)\, q^n$$

Proof of Eureka Thm

LHS rewrite

$$\left(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\right) q^{st+rt+rt} x^r y^s z^t$$

similar to Jacobi's four square thm

we want to rewrite

$$\sum_{r,s,t\geq 0} = 3\sum_{\substack{r=0\\ s,t\geq 1}} + 3\sum_{\substack{r,s=0\\ t\geq 1}} + \sum_{r,s,t=0} + \sum_{r,s,t>0}$$

so LHS when $x=y=z$

$$= 3\sum_{r,s\geq 1} q^{rs} x^{r+s} + 3\sum_{r\geq 1} x^r + 1$$

$$+ \left(\sum_{r,s,t>0} + \sum_{r,s,t<0}\right) q^{st+rs+rt} \cdot x^{r+s+t}$$

Proof of Eureka

Letting $q \mapsto q^2$, $x \to q$

The LHS is

$$1 + 3\sum_{r\geq 1} q^r + \sum_{r,s\geq 1} q^{2rs+r+s}$$

$$+\left(\sum_{r,s,t>0} + \sum_{r,s,t<0}\right) q^{2rs+2st+2rt} \cdot q^{r+s+t}$$

$$= \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3} = \sum_{n=0}^{\infty} r_{3\Delta}(n)$$

So tells us $r_{3\Delta}(n) \geq 1$ all n.

Summary

<u>last time</u>

- introduced a <u>heuristic</u>

$$m(x,q,z) \sim \sum_{r\geq 0} (-1)^r x^r q^{-\binom{r+1}{2}}$$

- new $m(x,q,z)$ identities
- express Hecke-type double-sums in terms of $m(x,q,z)$ fns.

<u>This time</u>

- Kronecker's identity
  applications: Lagranges four sq thm
  & Fermat's two sqr thm
- Heuristic & double sum analog of Kronecker's identity
  applications: Legendre | Gauss and Gauss's Eureka Thm.

next two lectures:

- Three field identities

- simultaneous representations of primes by binary quadratic forms

- a function from the lost notebook studied by Andrews, Dyson, and Hickerson

- quantum modular forms