THE SCHMIDT SUBSPACES OF HANKEL OPERATORS

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Abstract. These are (slightly expanded) lecture notes from a zoom mini-course given at the EIMI thematic program on Spectral Theory and Mathematical Physics, 3–5 November 2020. The content is based on the author's joint work with Patrick Gérard.

1. Introduction

1.1. Hankel and Toeplitz matrices. The most elementary way of approaching the definition of Hankel and Toeplitz operators is to consider them as infinite matrices in $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, with the following structure:

$$\Gamma = \{a(n+m)\}_{n,m=0}^{\infty}, \qquad T = \{a(n-m)\}_{n,m=0}^{\infty},$$

where $\{a(n)\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers. The aim of this mini-course is to give a description of eigenspaces of Toeplitz operators and Schmidt subspaces of Hankel operators.

1.2. Eigenspaces and Schmidt subspaces. Let A be a bounded operator in a Hilbert space. One says that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A, if the *eigenspace*

$$\mathrm{Ker}(A - \lambda I)$$

is non-trivial: $\mathrm{Ker}(A - \lambda I) \neq \{0\}$.

One says that $s > 0$ is a *singular value* of A, if the *Schmidt subspace*

$$E_A(s) := \mathrm{Ker}(A^*A - s^2 I)$$

is non-trivial. In other words, this means that there exists a non-zero pair (ξ, η) (called the *Schmidt pair*) of elements in our Hilbert space such that $A\xi = s\eta$ and $A^*\eta = s\xi$.

It is straightforward to see that A maps the Schmidt subspace $E_A(s)$ onto $E_{A^*}(s)$:

$$A : E_A(s) \to E_{A^*}(s), \quad A^* : E_A(s) \to E_{A^*}(s).$$

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1.3. **Hardy space.** We use the standard notation for the unit circle and the unit disk in $\mathbb{C}$:

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

There are three ways of thinking of the Hardy space:

- As $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$; in this case the elements are sequences
$$x = \{x_n\}_{n=0}^{\infty},$$
with $\sum_{n=0}^{\infty} |x_n|^2 < \infty$.
- As $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$, a subspace of L^2 on the unit circle; in this case the elements are the functions
$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, \quad |z| = 1$$
with the Fourier coefficients satisfying $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$.
- As $H^2(\mathbb{D})$, the space of analytic functions on the open unit disk; in this case, the elements are
$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, \quad |z| < 1$$
with the Taylor coefficients satisfying $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$.

The unitary map between the first and the second representation is effected by the Fourier series expansion.

The map from the second to the third representation is given by the analytic continuation inside the unit disk. The map from the third to the second representation is given by taking the boundary values

$$f(e^{i\theta}) = \lim_{r \to 1_-} f(re^{i\theta}), \qquad \text{a.e. } \theta \in (0, 2\pi).$$

We refer e.g. to [12, 15] for the theory of boundary behaviour of holomorphic functions in the unit disk.

The second and third representations are often used interchangeably. We will denote the space simply by H^2. We will denote by $\langle f, g \rangle$ the inner product of f and g in H^2, and we will denote by $\mathbb{1}$ the function in H^2 which is identically equal to 1.

Consider the space $H^2(\mathbb{D})$. We have $\langle f, \mathbb{1} \rangle = f(0)$ for any $f \in H^2(\mathbb{D})$. More generally, if $u_\alpha(z) = \dfrac{1}{1 - \overline{\alpha} z}$, where $|\alpha| < 1$, then u_α is the reproducing kernel of H^2, i.e.

$$\langle f, u_\alpha \rangle = f(\alpha), \quad f \in H^2.$$

The Szegő projection P is the orthogonal projection in $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$, given by

$$P : \sum_{n=-\infty}^{\infty} \widehat{f}(n) z^n \mapsto \sum_{n=0}^{\infty} \widehat{f}(n) z^n.$$

The shift operator S in H^2 is defined by $Sf(z) = zf(z)$, and its adjoint S^* in H^2 is given by

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

We will also need the Hardy space H^∞, which can be defined as $H^2 \cap L^\infty$ (or as the space of all bounded analytic functions in the unit disk).

1.4. **Toeplitz operators in Hardy space.** Let $a \in L^\infty(\mathbb{T})$; consider the operator T_a in H^2, given by

$$T_a f = P(a \cdot f), \quad f \in H^2.$$

The function a is called the symbol of T_a. It is straightforward to see that T_a is bounded and that the matrix of T_a in the standard basis $\{z^n\}_{n=0}^\infty$ is a Toeplitz matrix:

$$\langle T_a z^n, z^m \rangle = \langle P(az^n), z^m) \rangle = \langle az^n, z^m \rangle = \langle a, z^{m-n} \rangle = \widehat{a}(m-n), \quad n, m \geq 0.$$

Conversely, it is not difficult to prove that any bounded Toeplitz matrix

$$T = \{t(m-n)\}_{n,m=0}^\infty \quad \text{on } \ell^2(\mathbb{Z}_+)$$

is unitarily equivalent to T_a with some $a \in L^\infty$ (we will say that T is *realised* as T_a in the Hardy space). Moreover, we have the norm equality

$$\|T_a\| = \|a\|_{L^\infty}.$$

Toeplitz operators satisfy the commutation relation

$$S^* T_a S = T_a;$$

the proof of this is a simple exercise. In fact, if a bounded operator T_a satisfies this commutation relation, then it is necessarily a Toeplitz operator (this is called the Brown-Halmos theorem).

Example. We have $T_z = S$ and $T_{\overline{z}} = S^*$.

We refer, e.g. to [14] for background information on Toeplitz operators.

It is easy to see that T_a is compact if and only if $T_a = 0$. For a general symbol $a \in L^\infty$, the description of the spectrum of T_a is a difficult problem. Here we briefly mention the following facts for continuous symbols $a \in C(\mathbb{T})$:

- The essential spectrum of T_a is given by $\sigma_{\text{ess}}(T_a) = a(\mathbb{T})$.
- If $\lambda \notin a(\mathbb{T})$, then the winding number $\text{wind}(a - \lambda)$ around zero is well defined. The kernel of T_a is not empty if and only if this winding number is negative. In fact, $\dim \operatorname{Ker}(T_a - \lambda I) = -\text{wind}(a - \lambda)$.

1.5. **Hankel operators in Hardy space.** Let $a \in L^\infty(\mathbb{T})$; consider the operator Γ_a on H^2, given by

$$\Gamma_a f = P(a \cdot Jf), \quad Jf(z) = f(\overline{z}), \quad f \in H^2.$$

It is straightforward to see that the matrix of Γ_a in the standard basis of H^2 is a Hankel matrix:

$$\langle \Gamma_a z^n, z^m \rangle = \langle P(a\overline{z}^n), z^m \rangle = \langle a, z^{m+n} \rangle = \widehat{a}(n+m), \quad n, m \geq 0.$$

It is an easy exercise to check the commutation relation

$$S^* \Gamma_a = \Gamma_a S.$$

In fact, it is not difficult to show that any bounded operator that satisfies this relation is a Hankel operator.

Observe that Γ_a depends only on the Fourier coefficients $\widehat{a}(n)$ with $n \geq 0$. In other words, the symbol a is not uniquely defined: there are many symbols $b \neq a$ (they all differ by Fourier coefficients with $n < 0$) such that $\Gamma_a = \Gamma_b$; in particular, $\Gamma_a = \Gamma_{Pa}$.

It is clear from the definition that

$$\|\Gamma_a\| \leq \|a\|_{L^\infty}.$$

In contrast to Toeplitz operators, this is not an equality! To understand this, recall that there are many symbols b with $\Gamma_b = \Gamma_a$. Thus, we have

$$\|\Gamma_a\| \leq \inf\{\|b\|_{L^\infty} : \Gamma_b = \Gamma_a\}.$$

It turns out that here we have an equality; this is known as Nehari's theorem and it marks the start of the development of the modern theory of Hankel operators.

Theorem (Z. Nehari, 1957 [13])**.**

$$\|\Gamma_a\| = \inf\{\|b\|_{L^\infty} : \Gamma_b = \Gamma_a\},$$

and the infimum is attained on some $b \in L^\infty$.

Example. Let $u_\alpha(z) = \dfrac{1}{1 - \overline{\alpha} z}$, where $|\alpha| < 1$. Recall that u_α is the reproducing kernel of H^2, i.e.

$$\langle f, u_\alpha \rangle = f(\alpha), \quad f \in H^2.$$

It is a simple exercise to check that

$$\Gamma_{u_\alpha} = \langle \cdot, u_{\overline{\alpha}} \rangle u_\alpha,$$

i.e. Γ_{u_α} is a rank one Hankel operator.

Kronecker's theorem asserts that Γ_a is a finite rank operator if and only if Pa is a rational function with no poles in the closed unit disk. Using the above example, it is not difficult to prove one part of Kronecker's theorem. If Pa is a rational function, we can represent it as a sum of elementary fractions u_α and their derivatives. Each of them gives rise to a finite rank Hankel operator.

Combining Kronecker's theorem and Nehari's theorem, one can prove that Γ_a is compact iff $\Gamma_a = \Gamma_b$ for some $b \in C(\mathbb{T})$ (this is known as Hartman's theorem [9]).

We refer, e.g. to [14] and [16] for background information on Hankel operators.

Lastly, we would like to discuss one convention. If we want the symbol of Γ_a to be uniquely defined, we can require it to be analytic, i.e. $Pa = a$, or in other words $\widehat{a}(n) = 0$ for all $n < 0$. If we make this choice, then the symbol is given by $\Gamma_a \mathbb{1} = Pa = a$. In the later sections, we will make this choice. We note that if Γ_a is bounded, then $Pa = \Gamma_a \mathbb{1} \in H^2$.

2. Inner functions, model spaces and isometric multipliers

2.1. Inner functions. A non-constant function $\theta \in H^\infty$ is called *inner*, if $|\theta(z)| = 1$ for almost all z in the unit circle.

Example. Let $N \in \mathbb{N}$ and let $\{z_n\}_{n=1}^N$ be points in the open unit disk. Define

$$\theta(z) = \prod_{n=1}^N \frac{z_n - z}{1 - \overline{z_n} z}$$

for $|z| < 1$. Then θ is inner; it is a *Blaschke product of degree* N with zeros $\{z_n\}_{n=1}^N$.

Example. The previous example can be modified to the case of infinitely many zeros. The only new aspect is that one has to take care about the convergence of the infinite product. Let $\{z_n\}_{n=1}^\infty$ be points in the open unit disk, satisfying the condition

$$\sum_{n=1}^\infty (1 - |z_n|) < \infty.$$

Define

$$\theta(z) = \prod_{n=1}^\infty \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}$$

for $|z| < 1$. (The terms $\overline{z_n}/|z_n|$ are inserted in order to make the infinite product converge.) Then θ is inner; it is an infinite Blaschke product. We define the degree of θ to be infinity.

Example. Let $\mu \geq 0$ be a finite singular measure on the unit circle $\mathbb{T}$; define

$$\theta(z) = \exp\left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

for $|z| < 1$. Then θ is inner; it is a *singular inner function*; by definition, the degree of θ is infinity. For example, if μ is a point mass at 1 with $\mu(\{1\}) = c > 0$, we have

$$\theta(z) = \exp\left(c \frac{z+1}{z-1} \right).$$

In fact, every inner function can be represented as a product of a Blaschke product and a singular inner function.

2.2. **Beurling's theorem.** What are the invariant subspaces of S on H^2?

Let $\alpha_1, \dots, \alpha_n \in \mathbb{D}$, and let $M \subset H^2$ be a subspace defined by the condition $f(\alpha_1) = \cdots = f(\alpha_n) = 0$. It is clear that $SM \subset M$. Roughly speaking, Beurling's theorem says that a suitable generalisation of this example yields all invariant subspaces of S.

Theorem (A.Beurling (1949)). *Let $M \subset H^2$ be a closed subspace, $M \neq \{0\}$ and $M \neq H^2$. Suppose that M is invariant under the shift operator: $SM \subset M$. Then there exists an inner function θ such that*

$$M = \theta H^2 := \{\theta f : f \in H^2\}.$$

Of course, the converse is also true: every subspace θH^2 is invariant for S.

2.3. **Model spaces.** Observing that $SM \subset M$ if and only if $S^* M^\perp \subset M^\perp$, we obtain

Corollary. *Let $M \subset H^2$ be a closed subspace, $M \neq \{0\}$ and $M \neq H^2$. Suppose that M is invariant under the backwards shift operator: $S^* M \subset M$. Then there exists an inner function θ such that*

$$M = K_\theta := H^2 \cap (\theta H^2)^\perp.$$

The space K_θ is called a *model space*.

The name comes from the fact that $S^*|_{K_\theta}$ serves as a model for contractions from a certain class.

Let us rewrite the condition $f \in K_\theta$ in an equivalent way:
$f \perp \theta H^2 \Leftrightarrow \overline{\theta} f \perp H^2 \Leftrightarrow \overline{z}\theta\overline{f} \in H^2$.

Example. Let $\theta(z) = z^N$, $N \in \mathbb{N}$. Then

$$K_{z^N} = \{a_0 + \cdots + a_{N-1} z^{N-1} : a_0, \dots, a_{N-1} \in \mathbb{C}\}$$

is the space of all polynomials of degree $\leq N - 1$.

Example. Let $N \in \mathbb{N}$ and let θ be a Blaschke product of degree N with distinct zeros $\{z_n\}_{n=1}^N$. It is easy to see that

$$K_\theta = \operatorname{span}\{u_{z_n}\}_{n=1}^N, \quad u_\alpha(z) = \frac{1}{1 - \overline{\alpha} z}.$$

Indeed, recalling that u_α is the reproducing kernel, we see that the orthogonal complement to $\operatorname{span}\{u_{z_n}\}_{n=1}^N$ is precisely the linear subspace of functions $f \in H^2$ that vanish at all points $\{z_n\}_{n=1}^N$. This subspace coincides with θH^2.

We mention in passing a few properties of model spaces:

- $\theta(0) = 0$ if and only if $\mathbb{1} \in K_\theta$.
- The map $f \mapsto \overline{z}\theta\overline{f}$ is an involution on K_θ.
- We have $K_{\theta_1} \subset K_{\theta_2}$ if and only if θ_1 divides θ_2, i.e. if $\theta_2 = \theta_1 \varphi$ with some inner φ.

- Suppose for simplicity that the degree of θ is finite. Then the spectrum of $S^*|_{K_\theta}$ coincides (up to complex conjugation) with the set of zeros of θ.

2.4. **Isometric multipliers on model spaces.** Let M be a closed subspace in H^2, and let p be an analytic function in the open unit disk. One says that p is an *isometric multiplier* on M, if for every $f \in M$, we have $pf \in H^2$ and

$$\|pf\| = \|f\|.$$

In this case we will denote

$$pM = \{pf : f \in M\}.$$

Clearly, pM is a closed subspace in H^2.

Remark. Observe that if $\mathbb{1} \in M$, then (taking $f = \mathbb{1}$) we have $p \in M$ and $\|p\| = 1$.

Exercise. Check that if $\mathbb{1} \in M$, then p is (up to a unimodular complex factor) the normalised projection of $\mathbb{1}$ onto the space pM.

The interest to isometric multipliers on model spaces arose due to a result by E.Hayashi from 1986, who showed that all Toeplitz kernels are of the form pK_θ. We will come back to this later.

D.Sarason has characterised all isometric multipliers on a given model space K_θ. Before stating his result, we mention in passing that if g is any element of H^2 of norm one, then it can be represented as

$$g(z) = \frac{a(z)}{1 - b(z)}, \quad |z| < 1,$$

where $a, b \in H^\infty$ is a pair of functions such that $|a|^2 + |b|^2 = 1$ almost everywhere on the unit circle.

Theorem (D. Sarason, 1988 [18])**.** *Let θ be an inner function with $\theta(0) = 0$, and let $p \in H^2$ be a function of norm one. Then p is an isometric multiplier on K_θ if and only if it can be represented as*

$$p(z) = \frac{a(z)}{1 - \theta(z)b(z)}, \quad |z| < 1,$$

where $a, b \in H^\infty$ is a pair of functions such that $|a|^2 + |b|^2 = 1$ almost everywhere on the unit circle.

Some ideas of the proof. We will prove only the easy part of the theorem (the "if" part) in the easy case when $|b| \le \text{const} < 1$. Let us write $|p|^2$ on the unit circle. By a simple algebra, we have

$$|p|^2 = \frac{|a|^2}{|1 - \theta b|^2} = \frac{1 - |b|^2}{|1 - \theta b|^2} = 1 + \frac{\theta b}{1 - \theta b} + \frac{\overline{\theta b}}{1 - \overline{\theta b}}.$$

Now let us multiply this by $|f|^2$ and write the result as

$$|p|^2|f|^2 = |f|^2 + \frac{bf}{1-\theta b}\theta\overline{f} + \frac{\overline{bf}}{1-\overline{\theta b}}\overline{\theta}f. \tag{2.1}$$

Consider the second term in the right hand side. By the assumption on b, the term

$$\frac{bf}{1-\theta b}$$

is an element in H^2. Further, since $f \in K_\theta$, we have $\overline{z}\theta\overline{f} \in H^2$, and so $\theta\overline{f}$ is an element in H^2 which vanishes at the origin. It follows that

$$\frac{bf}{1-\theta b}\theta\overline{f}$$

is a function in H^1 which vanishes at the origin. Thus, its integral over the unit circle vanishes. The same considerations apply to the last term in the right hand side of (2.1): its integral over the unit circle vanishes. So, integrating (2.1), we obtain

$$\int_{-\pi}^{\pi} |p(e^{it})|^2 |f(e^{it})|^2 \, dt = \int_{-\pi}^{\pi} |f(e^{it})|^2 \, dt,$$

which means precisely that $\|pf\| = \|f\|$. $\square$

2.5. **Frostman shifts.** Here we address the following question. Let $M = pK_\theta$, where p is an isometric multiplier on K_θ. Are the parameters p, θ unique in the representation $M = pK_\theta$?

It is clear that one can multiply both p and θ by unimodular complex numbers without changing the space pK_θ. It turns out that there is another natural family of transformations on p and θ that leaves the space pK_θ invariant. To begin, consider the example of the previous theorem with both a and b being constants. Changing notation slightly, we see that for every $|\alpha| < 1$, the function $\sqrt{1-|\alpha|^2}/(1-\overline{\alpha}\theta)$ is an isometric multiplier on K_θ.

Exercise. 1. Check that

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta} K_\theta \subset K_{\theta_\alpha}, \quad \theta_\alpha = \frac{\alpha - \theta}{1-\overline{\alpha}\theta}.$$

2. Check that in fact we have the equality

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta} K_\theta = K_{\theta_\alpha}.$$

Hint: use the fact that $(\theta_\alpha)_\alpha = \theta$ and

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta_\alpha} = \frac{1-\overline{\alpha}\theta}{\sqrt{1-|\alpha|^2}}.$$

We can rewrite the result of this exercise as follows:

$$K_\theta = g_\alpha K_{\theta_\alpha}, \quad g_\alpha = \frac{1-\overline{\alpha}\theta}{\sqrt{1-|\alpha|^2}},$$

and g_α is the isometric multiplier on K_{θ_α}. This transformation is called the *Frostman shift*. From here we see that if p is an isometric multiplier on K_θ, then the space pK_θ can be equivalently written as

$$pK_\theta = pg_\alpha K_{\theta_\alpha},$$

where pg_α is an isometric multiplier on K_{θ_α}.

In fact, the converse statement also holds, see Crofoot [1]. If

$$pK_\theta = \widetilde{p}K_{\widetilde{\theta}},$$

where p is an isometric multiplier on K_θ and $\widetilde{p}$ is an isometric multiplier on $K_{\widetilde{\theta}}$, then for some constants $|\alpha| < 1$, $|c_1| = 1$, $|c_2| = 1$ we have

$$\widetilde{\theta} = c_1\theta_\alpha, \quad \widetilde{p} = c_2 p g_\alpha.$$

Suppose we have a subspace of the form pK_θ. It is often convenient to perform a Frostman shift with $\alpha = \theta(0)$. Then $\theta_\alpha(0) = 0$ and we write our subspace in an equivalent form $\widetilde{p}K_{\widetilde{\theta}}$ with $\widetilde{\theta}(0) = 0$.

Also, $\theta(0) = 0$ is a convenient normalisation which fixes the choices of θ and p up to unimodular constant factors.

2.6. **Nearly invariant subspaces.** First, some heuristics. Let $p \in H^\infty$, and let T_p be the operator of multiplication by p in H^2. By a direct calculation, the commutator of T_p with S^* is a rank one operator:

$$S^*T_p - T_pS^* = (S^*p)\langle\cdot, \mathbb{1}\rangle.$$

How does S^* act on pK_θ? Using the above calculation, we find that for $f \in K_\theta$,

$$S^*(pf) = p(S^*f) + (S^*p)\langle f, \mathbb{1}\rangle.$$

Thus, pK_θ is invariant for S^*, up to restricting to the subspace of codimension one.

The following definition was introduced by D. Hitt in 1988, see [11]. A closed subspace $M \subset H^2$ is called *nearly S^*-invariant*, if

$$S^*(M \cap \mathbb{1}^\perp) \subset M.$$

In other words, we require that if $f \in M$ and $f(0) = 0$, then $f(z)/z \in M$.

Observe that if $M \neq \{0\}$ is nearly S^*-invariant, then $M \not\perp \mathbb{1}$. Indeed, if $M \perp \mathbb{1}$ and if $f \in M$, then after dividing by z a finite number of times, we must arrive at a function which does not vanish at the origin, which contradicts the assumption $M \perp \mathbb{1}$. Because of this simple observation, the condition that $M \not\perp \mathbb{1}$ is often included in the definition of nearly S^*-invariance.

Let $M = pN$, where $S^*N \subset N$ and $p(0) \neq 0$. We claim that pN is nearly S^*-invariant. Indeed, if $f \in N$ and $pf \in pN$, then $(pf)(0) = 0$ means $f(0) = 0$, and so $p(z)f(z)/z = p(z)(f(z)/z) \in M$, because $f(z)/z \in N$.

Hitt's theorem gives a converse.

Theorem (D. Hitt, 1988 [11]). *Every nearly S^*-invariant subspace M is of the form $M = pN$, where $S^*N \subset N$ and p is an isometric multiplier on N. Thus, we have two possibilities: (i) $M = pK_\theta$, where θ is inner and p is an isometric multiplier on K_θ; (ii) $M = pH^2$, where p is an inner function.*

First let us discuss some heuristics behind the proof.

Suppose $M = pK_\theta$; how to identify p and θ? Recall that, by a Frostman shift, we can choose θ such that $\theta(0) = 0$, and then $\mathbb{1} \in K_\theta$. Also recall that in this case, p is the normalised projection of $\mathbb{1}$ onto M. This is how the proof below starts.

Some ideas of the proof. 1) Let p be the normalised projection of $\mathbb{1}$ onto M. For $f \in M$, write

$$f = c_0 p + f_1, \quad f_1 \perp p.$$

Since both f and p are in M, we also have $f_1 \in M$. By orthogonality,

$$\|f\|^2 = |c_0|^2 + \|f_1\|^2.$$

Further, $f_1 \perp p$ means $f_1 \perp \mathbb{1}$ and so, by the nearly S^*-invariance, we have $S^* f_1 = f_1/z \in M$. For f_1/z we write again

$$f_1/z = c_1 p + f_2, \quad f_2 \perp p.$$

Then we get

$$\|f_1\|^2 = |c_1|^2 + \|f_2\|^2$$

and again $S^* f_2 \in M$. Continuing recursively, we get

$$f_n/z = c_n p + f_{n+1}, \quad f_{n+1} \perp p,$$

and

$$\|f_n\|^2 = |c_n|^2 + \|f_{n+1}\|^2.$$

Linking these equations together gives

$$f(z) = (c_0 + c_1 z + \cdots + c_n z^n)p(z) + z^n f_{n+1}(z)$$

and

$$\|f\|^2 = |c_0|^2 + \cdots + |c_n|^2 + \|f_{n+1}\|^2 \geq |c_0|^2 + \cdots + |c_n|^2.$$

Inspecting the Taylor series of f/p at zero, we find that

$$f(z)/p(z) = \sum_{n=0}^{\infty} c_n z^n$$

and

$$\|f/p\|^2 = \sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2.$$

So the operator $T_{1/p} : M \to H^2$ is a contraction.

2) Consider the set

$$M_0 = \{f \in M : \|f/p\| = \|f\|\}.$$

Exercise. Using the previous step of the proof, prove that M_0 is a linear (linearity is non-trivial!) closed subspace of M. Further, prove that $T_{1/p}(M_0)$ is S^*-invariant.

3) Using a separate clever calculation with reproducing kernels, Hitt shows that actually $M_0 = M$. So now we have $T_{1/p}M = N$, or $M = pN$, where p is an isometric multiplier on N. □

3. Back to Toeplitz and Hankel operators

3.1. **Toeplitz kernels.** As already mentioned, Toeplitz operators satisfy the key commutation relation

$$S^* T_a S = T_a. \tag{3.1}$$

Here we determine the structure of Toeplitz kernels. First we make two remarks:

1) Since I is a Toeplitz operator with the symbol $\mathbb{1}$, we have

$$\operatorname{Ker}(T_a - \lambda I) = \operatorname{Ker} T_{a-\lambda\mathbb{1}}.$$

Thus, describing the structure of Toeplitz kernels is equivalent to describing the structure of all Toeplitz eigenspaces.

2) A deep theorem by M.Rosenblum [17] says that if T_a is a bounded self-adjoint Toeplitz operator with a non-constant symbol a, then the spectrum of T_a is purely absolutely continuous. In particular, T_a has no eigenvalues. This shows that the study of kernels of Toeplitz operators is a specifically non-selfadjoint problem.

Theorem (E. Hayashi, 1986 [10])**.** *Let T be a bounded Toeplitz operator in H^2 with a non-trivial kernel. Then there exists an inner function θ and an isometric multiplier p on K_θ such that*

$$\operatorname{Ker} T = pK_\theta.$$

Example. If θ is inner, then $\operatorname{Ker} T_{\overline{\theta}} = K_\theta$, i.e. in this case $p = 1$.

Proof. 1) Let us check that $\operatorname{Ker} T_a$ is nearly S^*-invariant. Suppose $T_a f = 0$ and $f \perp \mathbb{1}$; then $f = SS^* f$. Let us apply the commutation relation (3.1) to $S^* f$: we get

$$S^* T_a S S^* f = T_a S^* f.$$

The left hand side is $S^* T_a f = 0$, hence $S^* f \in \operatorname{Ker} T_a$, as claimed.

2) Since $\operatorname{Ker} T_a$ is a nearly invariant subspace, by Hitt's theorem there are two possibilities: (i) $\operatorname{Ker} T_a = pK_\theta$ where p is an isometric multiplier on K_θ, and (ii) $\operatorname{Ker} T_a = pH^2$, where p is inner. Let us show that the second possibility implies $T_a = 0$. Let $f \in H^2$, and $T_a(pf) = 0$, where p is inner. This means $P(apf) = 0$, i.e.

$$apf \in \overline{z}\overline{H^2}.$$

Since p is inner, this can be equivalently rewritten as

$$af \in \overline{pz}\,\overline{H^2}.$$

Since $\overline{p}\overline{H^2} \subset \overline{H^2}$, we obtain $f \in \operatorname{Ker} T_a$. Recall that f was an arbitrary element in H^2; this, we get $T_a = 0$. $\square$

Remark. Part 2) of the proof can be easily modified to show that in the representation $\operatorname{Ker} T_a = pK_\theta$, the isometric multiplier p cannot have any inner divisors (i.e. p is *outer*).

3.2. **Anti-linear representation for Hankel operators.** Recall that we have defined Hankel operators on H^2 as $\Gamma_a f = P(af(\overline{z}))$. Now it is convenient to switch to the following *anti-linear* version of Hankel operators:

$$H_a f = P(a\overline{f}), \quad f \in H^2.$$

Observe that for $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$, we have

$$\overline{f(z)} = \sum_{n=0}^\infty \overline{\widehat{f}(n)} z^{-n}, \qquad f(\overline{z}) = \sum_{n=0}^\infty \widehat{f}(n) z^{-n},$$

and so the operators Γ_a and H_a are related through the anti-linear involution:

$$H_a = \Gamma_a \mathcal{C}, \quad \mathcal{C} : \sum_{n=0}^\infty \widehat{f}(n) z^n \mapsto \sum_{n=0}^\infty \overline{\widehat{f}(n)} z^n.$$

It is easy to see that $\Gamma_a \mathcal{C} = \mathcal{C} \Gamma_a^*$, and so

$$H_a^2 = \Gamma_a \mathcal{C} \Gamma_a \mathcal{C} = \Gamma_a \Gamma_a^*.$$

Thus, any Schmidt subspace $E_{\Gamma_a^*}(s)$ can be expressed as

$$E_{\Gamma_a^*}(s) = \operatorname{Ker}(H_a^2 - s^2 I).$$

It follows that H_a acts on $\operatorname{Ker}(H_a^2 - s^2 I)$. This is one of the advantages of the anti-linear representation: H_a acts on $E_{\Gamma_a^*}(s)$, while Γ_a acts from $E_{\Gamma_a}(s)$ to $E_{\Gamma_a^*}(s)$.

Notation: from now on, we will assume that the symbol of a Hankel operator is analytic and denote it by u. Recall again that $u = H_u \mathbb{1} \in H^2$.

3.3. **Hankel kernels and ranges.** Recall that Hankel operators satisfy the commutation relation

$$S^* H_u = H_u S.$$

First we discuss Hankel kernels. If $H_u f = 0$, then by the above commutation relation we also have

$$0 = S^* H_u f = H_u S f.$$

Thus, Hankel kernels are invariant under the shift operator S, and so by Beurling's theorem they have the form ψH^2 for some inner function ψ. Taking orthogonal complements, we obtain

$$\overline{\operatorname{Ran} H_u} = K_\psi.$$

Example. Let θ be inner; consider the Hankel operator H_θ,

$$H_\theta f = P(\theta \overline{f}).$$

It is straightforward to see that in this case we have

$$\operatorname{Ker} H_\theta = z\theta H^2, \quad \operatorname{Ran} H_\theta = K_{z\theta},$$

and H_θ is an anti-linear involution on $K_{z\theta}$,

$$H_\theta f = \theta \overline{f}, \quad f \in K_{z\theta}.$$

It follows that H_θ^2 is the orthogonal projection onto $K_{z\theta}$. In other words, in this case we have only one singular value $s = 1$, and the corresponding Schmidt subspace is $E_{H_\theta}(1) = K_{z\theta}$.

3.4. **Schmidt subspaces of Hankel operators.**

Theorem (Gerard-Pushnitski [6, 8]). *Let H_u be a bounded Hankel operator on H^2, and let $s > 0$ be a singular value of H_u:*

$$E_{H_u}(s) = \operatorname{Ker}(H_u^2 - s^2 I) \neq \{0\}.$$

Then there exists an inner function θ and an isometric multiplier p on K_θ such that

$$E_{H_u}(s) = pK_\theta.$$

Before embarking on the proof, we note that $E_{H_u}(s)$ may or may not be nearly S^*-invariant. Indeed, it may happen that $E_{H_u}(s)$ is orthogonal to $\mathbb{1}$.

Example. Let $0 < \alpha < 1$, and let

$$u(z) = \frac{1-\alpha^2}{1-\alpha z^2}.$$

Exercise. Check that $H_u u = u$.

Thus, 1 is a singular value, and $u \in E_{H_u}(1)$.

Exercise. Applying S^* to the identity $H_u u = u$, check that $H_u(zu) = \alpha(zu)$.

Thus, α is a singular value, and $zu \in E_{H_u}(\alpha)$.

Exercise. Check that $\operatorname{rank} H_u = 2$. Deduce that

$$E_{H_u}(1) = \operatorname{span}\{u\}, \qquad E_{H_u}(\alpha) = \operatorname{span}\{zu\}.$$

Summarising, we see that $E_{H_u}(\alpha) \perp \mathbb{1}$.

Proof of the Theorem in the case $E_{H_u}(s) \not\perp \mathbb{1}$.

1) Let us establish some identities. We need the rank one identity

$$SS^* = I - \langle \cdot, \mathbb{1} \rangle,$$

and the obvious identity $H_u\mathbb{1} = u$. Using the commutation relation $S^*H_u = H_uS$, we have

$$S^*H_u^2S = H_uSS^*H_u = H_u^2 - \langle\cdot, H_u\mathbb{1}\rangle H_u\mathbb{1} = H_u^2 - \langle\cdot, u\rangle u.$$

(Compare this with the identity $S^*TS = T$ for Toeplitz operators!) Let us multiply the last identity by S^* on the right. After rearranging, we obtain

$$S^*H_u^2 - H_u^2S^* = \langle\cdot, \mathbb{1}\rangle S^*H_u u - \langle\cdot, Su\rangle u. \tag{3.2}$$

2) We need to establish the existence of an element $g \in E_{H_u}(s)$ such that $\langle u, g\rangle \neq 0$. This follows from the assumption $E_{H_u}(s) \not\perp \mathbb{1}$. Indeed, let $h \in E_{H_u}(s)$ be such that $\langle h, \mathbb{1}\rangle \neq 0$; take $g = H_uh$. Then

$$\langle u, g\rangle = \langle u\overline{g}, \mathbb{1}\rangle = \langle H_ug, \mathbb{1}\rangle = \langle H_u^2h, \mathbb{1}\rangle = s^2\langle h, \mathbb{1}\rangle \neq 0.$$

3) Let us prove that $E_{H_u}(s)$ is nearly S^*-invariant. Let $f \in E_{H_u}(s) \cap \mathbb{1}^\perp$, and let g as above. Let us take the bilinear form of (3.2) on the elements f, g. For the left hand side, we have

$$\langle S^*H_u^2f, g\rangle - \langle H_u^2S^*f, g\rangle = s^2\langle S^*f, g\rangle - \langle S^*f, H_u^2g\rangle = s^2\langle S^*f, g\rangle - s^2\langle S^*f, g\rangle = 0.$$

For the right hand side, we have

$$\langle f, \mathbb{1}\rangle\langle S^*H_uu, g\rangle - \langle f, Su\rangle\langle u, g\rangle;$$

by assumption $f \perp \mathbb{1}$, and so we obtain

$$\langle f, Su\rangle\langle u, g\rangle = 0.$$

But $\langle u, g\rangle \neq 0$, and so we obtain that $\langle f, Su\rangle = 0$.

Now let us substitute f back into (3.2) and use the latter fact; the right hand side vanishes and we have

$$S^*H_u^2f - H_u^2S^*f = 0.$$

Since $H_u^2f = s^2f$, this can be rewritten as

$$(H_u^2 - s^2I)S^*f = 0,$$

and so $S^*f \in E_{H_u}(s)$, as claimed.

Thus, $E_{H_u}(s)$ is a nearly S^*-invariant subspace.

4) By Hitt's theorem, either $E_{H_u}(s) = pK_\theta$ (where p is an isometric multiplier on K_θ) or $E_{H_u}(s) = pH^2$ (where p is inner).

Exercise. Show that the second option is not possible. Use the fact that the calculation from the previous step of the proof shows that

$$S^*(E_{H_u}(s) \cap \mathbb{1}^\perp) \subset E_{H_u}(s) \cap u^\perp.$$

Compare this with

$$S^*(pH^2 \cap \mathbb{1}^\perp) = pH^2$$

if p is inner. Bring this to a contradiction.

This completes the proof. $\square$

Proof of the Theorem in case $E_{H_u}(s) \perp \mathbb{1}$. Let α, $|\alpha| < 1$, be such that α is not a common zero of all elements of $E_{H_u}(s)$. Let μ be the Moebius map

$$\mu(z) = \frac{\alpha - z}{1 - \overline{\alpha} z},$$

mapping the unit disk onto itself, and let U_μ be the corresponding unitary operator on H^2:

$$U_\mu f(z) = \frac{\sqrt{1 - |\alpha|^2}}{1 - \overline{\alpha} z} f(\mu(z)).$$

By a direct calculation, U_μ is a unitary involution on H^2.

Exercise. Check that $U_\mu H_u U_\mu = H_w$ with some symbol w.

Now consider $M = U_\mu E_{H_u}(s)$. Then $M = E_{H_w}(s)$ and $z = 0$ is not a common zero of all elements of M, i.e. $M \not\perp \mathbb{1}$. By the previous part of the proof, it follows that $M = pK_\theta$ with some θ, and p is an isometric multiplier on K_θ. Now

$$E_{H_u}(s) = U_\mu(pK_\theta).$$

Exercise. Check that

$$U_\mu(pK_\theta) = (p \circ \mu) K_{\theta \circ \mu},$$

and $p \circ \mu$ is an isometric multiplier on $K_{\theta \circ \mu}$.

This completes the proof. □

3.5. **The cubic Szegő equation and inverse spectral problems.** The motivation for the study of Schmidt subspaces of Hankel operators comes from the work of Patrick Gérard and Sandrine Grellier in 2010–2014. In [2], they introduced the *cubic Szegő equation*

$$i\frac{\partial u}{\partial t} = P(|u|^2 u), \quad u = u(z;t), \quad z \in \mathbb{T}, \quad t \in \mathbb{R},$$

as a model for totally non-dispersive evolution equations. Here for each $t \in \mathbb{R}$, the function $u(\cdot, t)$ is an element of the Hardy class $H^2 = H^2(\mathbb{T})$ and $P : L^2 \to H^2$ is the usual Szegő projection. It turned out [2, 3] that this equation is completely integrable and possesses a Lax pair. Indeed, a function u is a solution to the cubic Szegő equation if and only if

$$\frac{d}{dt} H_u = [B_u, H_u],$$

where H_u is an anti-linear Hankel operator with the symbol u as discussed above, and B_u is a certain auxiliarly skew-selfadjoint operator. In particular, it follows that if the operator H_u is compact, then its singular values are integrals of motion for the cubic Szegő equation.

In order to solve the Cauchy problem for the cubic Szegő equation, one must therefore develop a version of direct and inverse spectral theory for H_u. The spectral data in this problem involves the sequence of singular values of H_u and the sequence of inner functions, parameterising the Schmidt subspaces of H_u. This was achieved in [4, 5] for $u \in \mathrm{VMOA}(\mathbb{T})$, which corresponds to *compact* Hankel operators H_u.

Without going into details, we mention the main points of this construction (see also [7]). For simplicity of discussion, let us assume that H_u is a finite rank operator, i.e. u is a rational function analytic in the unit disk, with no poles on the unit circle. Denote by $\{s_n\}_{n=1}^N$ the sequence of singular values of H_u, enumerated in the decreasing order; for each s_n, we have a non-trivial Schmidt subspace

$$E_{H_u}(s_n) = \operatorname{Ker}(H_u^2 - s_n^2 I).$$

(Note that usually singular values are enumerated with multiplicities, but here we list every distinct singular value exactly once!) Further, by our main result, each of these Schmidt subspaces can be written as

$$E_{H_u}(s_n) = p_n K_{\theta_n}.$$

In general, the parameters p_n and θ_n are not unique. But my making a Frostman shift we can ensure that $\theta_n(0) = 0$ for all n, and then they are unique up to a unimodular multiplicative factor. Part of our spectral data for H_u will be the pair

$$(\{s_n\}_{n=1}^N, \{\theta_n\}_{n=1}^N).$$

It turns out that this data is insufficient to recover the symbol H_u; we need to complement it by the spectral data corresponding to the auxiliary finite rank Hankel operator H_{S^*u}. Let $\{\widetilde{s}_n\}_{n=1}^N$ be the singular values of H_{S^*u} and let

$$E_{H_{S^*u}}(\widetilde{s}_n) = \widetilde{p}_n K_{\widetilde{\theta}_n}$$

be the corresponding Schmidt subspaces, where the parameters are normalised by $\widetilde{\theta}_n(0) = 0$. Then it turns out that the spectral data

$$(\{s_n\}_{n=1}^N, \{\widetilde{s}_n\}_{n=1}^N, \{\theta_n\}_{n=1}^N, \{\widetilde{\theta}_n\}_{n=1}^N)$$

uniquely determines the symbol u. Moreover, for any sets of distinct positive numbers $\{s_n\}_{n=1}^N$, $\{\widetilde{s}_n\}_{n=1}^N$ (we need a certain interlacing condition to hold) and any collection of inner functions θ_n, $\widetilde{\theta}_n$, one can construct a unique symbol u such that the Hankel operator H_u has the corresponding spectral data.

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