Recent analytic and spectral results for the multi-particle Schrödinger operator: Lecture 2: compact operators

Alexander Sobolev

University College London

St Petersburg, November 2020

Classes of compact operators

Let $\mathcal{H}$ and $\mathcal{G}$ be separable Hilbert spaces. Let $T : \mathcal{H} \to \mathcal{G}$ be a compact operator. If $\mathcal{H} = \mathcal{G}$ and $T = T^* \geq 0$, then $\lambda_k(T)$, $k = 1, 2, \dots,$ are the positive eigenvalues of T numbered in descending order counting multiplicity. For arbitrary $\mathcal{H}$, $\mathcal{G}$ and compact T, let $s_k(T) = \sqrt{\lambda_k(T^*T)} = \sqrt{\lambda_k(TT^*)}, \quad k = 1, 2, \dots,$ be the *singular values* of T.

Note the useful inequalities

$$s_{2k}(T_1 + T_2) \leq s_{2k-1}(T_1 + T_2) \leq s_k(T_1) + s_k(T_2),$$

$$\sum_{k=1}^{r} s_k(T_1 + T_2) \leq \sum_{k=1}^{r} s_k(T_1) + \sum_{k=1}^{r} s_k(T_2), \quad \forall r,$$

that hold for any two compact T_1, T_2.

Classes of compact operators

Let $\mathcal{H}$ and $\mathcal{G}$ be separable Hilbert spaces. Let $T : \mathcal{H} \to \mathcal{G}$ be a compact operator. If $\mathcal{H} = \mathcal{G}$ and $T = T^* \ge 0$, then $\lambda_k(T)$, $k = 1, 2, \dots$, are the positive eigenvalues of T numbered in descending order counting multiplicity. For arbitrary $\mathcal{H}$, $\mathcal{G}$ and compact T, let $s_k(T) = \sqrt{\lambda_k(T^*T)} = \sqrt{\lambda_k(TT^*)}, \quad k = 1, 2, \dots$, be the *singular values* of T.

Note the useful inequalities

$$s_{2k}(T_1 + T_2) \le s_{2k-1}(T_1 + T_2) \le s_k(T_1) + s_k(T_2),$$

$$\sum_{k=1}^{r} s_k(T_1 + T_2) \le \sum_{k=1}^{r} s_k(T_1) + \sum_{k=1}^{r} s_k(T_2), \quad \forall r,$$

that hold for any two compact T_1, T_2.

More generally: Rotfeld's Theorem. For any concave non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$, s.t. $\Phi(0) = 0$:

$$\sum_{k=1}^{r} \Phi\big(s_k(T_1 + T_2)\big) \le \sum_{k=1}^{r} \Phi\big(s_k(T_1)\big) + \sum_{k=1}^{r} \Phi\big(s_k(T_2)\big), \quad \forall r \ge 1.$$

Schatten-von Neumann classes $\mathbf{S}_p$, $0 < p < \infty$. The operator $T \in \mathbf{S}_p$ if

$$\|T\|_p = \left[\sum_{k=1}^{\infty} s_k(T)^p\right]^{\frac{1}{p}} < \infty.$$

It is a norm if $p \geq 1$. If $p < 1$, then by Rotfeld's Theorem

$$\|T_1 + T_2\|_p^p \leq \|T_1\|_p^p + \|T_2\|_p^p, \quad p\text{-triangle inequality.}$$

For $T \in \mathbf{S}_p$ we have $s_k(T) = o(k^{-1/p})$.

Schatten-von Neumann classes $\mathbf{S}_p$, $0 < p < \infty$. The operator $T \in \mathbf{S}_p$ if

$$\|T\|_p = \left[\sum_{k=1}^{\infty} s_k(T)^p\right]^{\frac{1}{p}} < \infty.$$

It is a norm if $p \geq 1$. If $p < 1$, then by Rotfeld's Theorem

$$\|T_1 + T_2\|_p^p \leq \|T_1\|_p^p + \|T_2\|_p^p, \quad p\text{-triangle inequality.}$$

For $T \in \mathbf{S}_p$ we have $s_k(T) = o(k^{-1/p})$.

Classes $\mathbf{S}_{p,\infty}$, $0 < p < \infty$.

$$\mathbf{S}_{p,\infty} = \{T \in \mathbf{S}_\infty : s_k(T) = O(k^{-1/p})\},$$
$$\mathbf{S}^\circ_{p,\infty} = \{T \in \mathbf{S}_\infty : s_k(T) = o(k^{-1/p})\}.$$

Denote

$$\|T\|_{p,\infty} = \sup_k k^{\frac{1}{p}} s_k(T).$$

The class $\mathbf{S}_{p,\infty}$ is a complete linear space with the quasi-norm $\|T\|_{p,\infty}$.

Schatten-von Neumann classes $\mathbf{S}_p$, $0 < p < \infty$. The operator $T \in \mathbf{S}_p$ if

$$\|T\|_p = \left[\sum_{k=1}^{\infty} s_k(T)^p\right]^{\frac{1}{p}} < \infty.$$

It is a norm if $p \geq 1$. If $p < 1$, then by Rotfeld's Theorem

$$\|T_1 + T_2\|_p^p \leq \|T_1\|_p^p + \|T_2\|_p^p, \quad p\text{-triangle inequality.}$$

For $T \in \mathbf{S}_p$ we have $s_k(T) = o(k^{-1/p})$.

Classes $\mathbf{S}_{p,\infty}$, $0 < p < \infty$.

$$\mathbf{S}_{p,\infty} = \{T \in \mathbf{S}_\infty : s_k(T) = O(k^{-1/p})\},$$
$$\mathbf{S}^\circ_{p,\infty} = \{T \in \mathbf{S}_\infty : s_k(T) = o(k^{-1/p})\}.$$

Denote

$$\|T\|_{p,\infty} = \sup_k k^{\frac{1}{p}} s_k(T).$$

The class $\mathbf{S}_{p,\infty}$ is a complete linear space with the quasi-norm $\|T\|_{p,\infty}$.

Proposition. For all $p > 0$:

$$\|T_1 + T_2\|_{p,\infty}^{\frac{p}{p+1}} \leq \|T_1\|_{p,\infty}^{\frac{p}{p+1}} + \|T_2\|_{p,\infty}^{\frac{p}{p+1}}.$$

If $p > 1$, then $\mathbf{S}_{p,\infty}$ has an equivalent norm:

Proposition $p > 1$**.** Let $T \in \mathbf{S}_{p,\infty}$, $p > 1$, and let

$$\langle T \rangle_p = \sup_{r\in\mathbb{N}} r^{1/p-1} \sum_{k=1}^{r} s_k(T).$$

Then $\langle T \rangle_p$ is a norm and

$$\|T\|_{p,\infty} \le \langle T \rangle_p \le p(p-1)^{-1}\|T\|_{p,\infty}.$$

If $p > 1$, then $\mathbf{S}_{p,\infty}$ has an equivalent norm:

Proposition $p > 1$. Let $T \in \mathbf{S}_{p,\infty}$, $p > 1$, and let

$$\langle T \rangle_p = \sup_{r \in \mathbb{N}} r^{1/p-1} \sum_{k=1}^{r} s_k(T).$$

Then $\langle T \rangle_p$ is a norm and

$$\|T\|_{p,\infty} \le \langle T \rangle_p \le p(p-1)^{-1} \|T\|_{p,\infty}.$$

Proof. The triangle inequality follows from

$$\sum_{k=1}^{r} s_k(T_1 + T_2) \le \sum_{k=1}^{r} s_k(T_1) + \sum_{k=1}^{r} s_k(T_2), \quad \forall r.$$

□

Proposition $p < 1$. Let $T \in \mathbf{S}_{p,\infty}$, $p < 1$, and let

$$\langle T \rangle_p = \sup_{t>0} t^{1-1/p} \left[\sum_{k=1}^{\infty} \min\{t, s_k(T)\} \right]^{\frac{1}{p}}.$$

Then $\| T \|_{p,\infty} \le \langle T \rangle_p \le (1-p)^{-1/p} \| T \|_{p,\infty}$, and

$$\langle T_1 + T_2 \rangle_p^p \le \langle T_1 \rangle_p^p + \langle T_2 \rangle_p^p.$$

Proposition $p < 1$. Let $T \in \mathbf{S}_{p,\infty}$, $p < 1$, and let

$$\langle T \rangle_p = \sup_{t>0} t^{1-1/p} \left[\sum_{k=1}^{\infty} \min\{t, s_k(T)\} \right]^{\frac{1}{p}}.$$

Then $\| T \|_{p,\infty} \le \langle T \rangle_p \le (1-p)^{-1/p} \| T \|_{p,\infty}$, and

$$\langle T_1 + T_2 \rangle_p^p \le \langle T_1 \rangle_p^p + \langle T_2 \rangle_p^p.$$

Proof. For the p-triangle inequality use Rotfeld's Theorem: for any concave non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$, s.t. $\Phi(0) = 0$:

$$\sum_{k=1}^{m} \Phi\big(s_k(T_1 + T_2)\big) \le \sum_{k=1}^{m} \Phi\big(s_k(T_1)\big) + \sum_{k=1}^{m} \Phi\big(s_k(T_2)\big), \quad \forall m \ge 1.$$

Use it with $\Phi_t(x) = t^{p-1} \min\{t, x\}$, $t > 0$. The p-triangle inequality follows. □

Proposition $p<1$. Let $T\in \mathbf{S}_{p,\infty}$, $p<1$, and let

$$\langle T\rangle_p = \sup_{t>0} t^{1-1/p}\left[\sum_{k=1}^\infty \min\{t, s_k(T)\}\right]^{\frac{1}{p}}.$$

Then $\|T\|_{p,\infty}\le \langle T\rangle_p \le (1-p)^{-1/p}\|T\|_{p,\infty}$, and

$$\langle T_1+T_2\rangle_p^p \le \langle T_1\rangle_p^p + \langle T_2\rangle_p^p.$$

Proof. For the p-triangle inequality use Rotfeld's Theorem: for any concave non-decreasing function $\Phi:[0,\infty)\to[0,\infty)$, s.t. $\Phi(0)=0$:

$$\sum_{k=1}^m \Phi\big(s_k(T_1+T_2)\big) \le \sum_{k=1}^m \Phi\big(s_k(T_1)\big) + \sum_{k=1}^m \Phi\big(s_k(T_2)\big),\quad \forall m\ge 1.$$

Use it with $\Phi_t(x) = t^{p-1}\min\{t,x\}$, $t>0$. The p-triangle inequality follows. □

Corollary.

$$\Big\|\sum_j T_j\Big\|_{p,\infty}^p \le (1-p)^{-1}\sum_j \|T_j\|_{p,\infty}^p,\quad p<1.$$

For $T \in \mathbf{S}_{p,\infty}$ the numbers

$$\begin{cases} \mathrm{G}_p(T) = \big(\limsup_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \\ \mathrm{g}_p(T) = \big(\liminf_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \end{cases}$$

are finite and satisfy the inequality

$$\mathrm{g}_p(T) \le \mathrm{G}_p(T) \le \|T\|_{p,\infty}^p.$$

For $T \in \mathbf{S}_{p,\infty}$ the numbers

$$\begin{cases} \mathrm{G}_p(T) = \big(\limsup_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \\ \mathrm{g}_p(T) = \big(\liminf_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \end{cases}$$

are finite and satisfy the inequality

$$\mathrm{g}_p(T) \le \mathrm{G}_p(T) \le \|T\|_{p,\infty}^p.$$

If $\mathrm{G}_p(T) = \mathrm{g}_p(T)$, then the singular values of T satisfy the asymptotic formula

$$s_n(T) = \big(\mathrm{G}_p(T)\big)^{\frac{1}{p}} n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}), \ n \to \infty.$$

For $T \in \mathbf{S}_{p,\infty}$ the numbers

$$\begin{cases} \mathrm{G}_p(T) = \big(\limsup_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \\ \mathrm{g}_p(T) = \big(\liminf_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \end{cases}$$

are finite and satisfy the inequality

$$\mathrm{g}_p(T) \le \mathrm{G}_p(T) \le \|T\|_{p,\infty}^p.$$

If $\mathrm{G}_p(T) = \mathrm{g}_p(T)$, then the singular values of T satisfy the asymptotic formula

$$s_n(T) = \big(\mathrm{G}_p(T)\big)^{\frac{1}{p}} n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}),\ n \to \infty.$$

Observe: $\mathrm{g}_p(TT^*) = \mathrm{g}_p(T^*T) = \mathrm{g}_{2p}(T),\ \mathrm{G}_p(TT^*) = \mathrm{G}_p(T^*T) = \mathrm{G}_{2p}(T)$.

For $T \in \mathbf{S}_{p,\infty}$ the numbers

$$\begin{cases} \mathrm{G}_p(T) = \big(\limsup_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \\ \mathrm{g}_p(T) = \big(\liminf_{k\to\infty} k^{\frac{1}{p}} s_k(T)\big)^p, \end{cases}$$

are finite and satisfy the inequality

$$\mathrm{g}_p(T) \le \mathrm{G}_p(T) \le \|T\|_{p,\infty}^p.$$

If $\mathrm{G}_p(T) = \mathrm{g}_p(T)$, then the singular values of T satisfy the asymptotic formula

$$s_n(T) = \big(\mathrm{G}_p(T)\big)^{\frac{1}{p}} n^{-\frac{1}{p}} + o(n^{-\frac{1}{p}}),\ n \to \infty.$$

Observe:$\mathrm{g}_p(TT^*) = \mathrm{g}_p(T^*T) = \mathrm{g}_{2p}(T),\ \mathrm{G}_p(TT^*) = \mathrm{G}_p(T^*T) = \mathrm{G}_{2p}(T)$.
Fact:

$$\mathrm{G}_p(T) = \inf_{R\in\mathbf{S}_{p,\infty}^\circ} \|T + R\|_{p,\infty}^p.$$

The functional $\mathrm{G}_p(T)$, $p < 1$, also satisfies the p-triangle inequality:

Proposition. Suppose that $T_j \in \mathbf{S}_{p,\infty}$, $j = 1, 2, \dots,$ with some $p < 1$ and that

$$\sum_j \|T_j\|_{p,\infty}^p < \infty.$$

Then

$$\mathrm{G}_p\Big(\sum_j T_j\Big) \le (1-p)^{-1} \sum_j \mathrm{G}_p(T_j).$$

The functionals G_p and g_p are continuous on $\mathbf{S}_{p,\infty}$:

Proposition. If $T_1, T_2 \in \mathbf{S}_{p,\infty}$, $0 < p < \infty$, then

$$\left|\mathrm{G}_p(T_1)^{\frac{1}{p+1}} - \mathrm{G}_p(T_2)^{\frac{1}{p+1}}\right| \le \mathrm{G}_p(T_1 - T_2)^{\frac{1}{p+1}},$$
$$\left|\mathrm{g}_p(T_1)^{\frac{1}{p+1}} - \mathrm{g}_p(T_2)^{\frac{1}{p+1}}\right| \le \mathrm{G}_p(T_1 - T_2)^{\frac{1}{p+1}}.$$

The functionals G_p and g_p are continuous on $\mathbf{S}_{p,\infty}$:

Proposition. If $T_1, T_2 \in \mathbf{S}_{p,\infty}$, $0 < p < \infty$, then

$$\big|\mathsf{G}_p(T_1)^{\frac{1}{p+1}} - \mathsf{G}_p(T_2)^{\frac{1}{p+1}}\big| \le \mathsf{G}_p(T_1 - T_2)^{\frac{1}{p+1}},$$
$$\big|\mathsf{g}_p(T_1)^{\frac{1}{p+1}} - \mathsf{g}_p(T_2)^{\frac{1}{p+1}}\big| \le \mathsf{G}_p(T_1 - T_2)^{\frac{1}{p+1}}.$$

Corollary. Suppose that $\mathsf{G}_p(T_1 - T_2) = 0$. Then $\mathsf{G}_p(T_1) = \mathsf{G}_p(T_2)$, $\mathsf{g}_p(T_1) = \mathsf{g}_p(T_2)$.

The functionals G_p and g_p are continuous on $\mathbf{S}_{p,\infty}$:

Proposition. If $T_1, T_2 \in \mathbf{S}_{p,\infty}$, $0 < p < \infty$, then

$$\left|\mathrm{G}_p(T_1)^{\frac{1}{p+1}} - \mathrm{G}_p(T_2)^{\frac{1}{p+1}}\right| \le \mathrm{G}_p(T_1 - T_2)^{\frac{1}{p+1}},$$
$$\left|\mathrm{g}_p(T_1)^{\frac{1}{p+1}} - \mathrm{g}_p(T_2)^{\frac{1}{p+1}}\right| \le \mathrm{G}_p(T_1 - T_2)^{\frac{1}{p+1}}.$$

Corollary. Suppose that $\mathrm{G}_p(T_1 - T_2) = 0$. Then $\mathrm{G}_p(T_1) = \mathrm{G}_p(T_2)$, $\mathrm{g}_p(T_1) = \mathrm{g}_p(T_2)$.

More generally:

Corollary. Suppose that $T \in \mathbf{S}_{p,\infty}$ and that for every $\nu > 0$ there exists an operator $T_\nu \in \mathbf{S}_{p,\infty}$ such that $\mathrm{G}_p(T - T_\nu) \to 0$, $\nu \to 0$. Then the functionals $\mathrm{G}_p(T_\nu), \mathrm{g}_p(T_\nu)$ have limits as $\nu \to 0$ and

$$\lim_{\nu\to 0} \mathrm{G}_p(T_\nu) = \mathrm{G}_p(T), \quad \lim_{\nu\to 0} \mathrm{g}_p(T_\nu) = \mathrm{g}_p(T).$$

Singular values of integral operators

M. Birman - M.Solomyak (1977). Let $\mathcal{C} = (0,1)^d$.

Proposition. Let $T_{ba} : \mathsf{L}^2(\Lambda) \to \mathsf{L}^2(\mathbb{R}^n)$, be the integral operator of the form

$$(T_{ba}u)(t) = b(t) \int T(t,x)a(x)u(x)\,dx,$$

where $a \in \mathsf{L}^2(\Lambda)$, $b \in \mathsf{L}^2_{\mathrm{loc}}(\mathbb{R}^n)$, and the kernel $T(t,x)$, $t \in \mathbb{R}^n$, $x \in \Lambda$, is such that $T(t,\ \cdot\ ) \in \mathsf{H}^l(\Lambda)$ with some $l = 1, 2, \dots,$ $2l > d$, a.e. $t \in \mathbb{R}^n$.

Singular values of integral operators

M. Birman - M.Solomyak (1977). Let $\mathcal{C} = (0,1)^d$.

Proposition. Let $T_{ba} : \mathsf{L}^2(\Lambda) \to \mathsf{L}^2(\mathbb{R}^n)$, be the integral operator of the form

$$(T_{ba}u)(t) = b(t) \int T(t,x)a(x)u(x)\,dx,$$

where $a \in \mathsf{L}^2(\Lambda)$, $b \in \mathsf{L}^2_{\mathrm{loc}}(\mathbb{R}^n)$, and the kernel $T(t,x)$, $t \in \mathbb{R}^n$, $x \in \Lambda$, is such that $T(t,\ \cdot\ ) \in \mathsf{H}^l(\Lambda)$ with some $l = 1, 2, \dots,\ 2l > d$, a.e. $t \in \mathbb{R}^n$. Then

$$s_k(T_{ba}) \lesssim k^{-\frac{1}{2}-\frac{l}{d}} \left[\int \|T(t,\ \cdot\ )\|^2_{\mathsf{H}^l} |b(t)|^2\,dt\right]^{\frac{1}{2}} \|a\|_{\mathsf{L}^2(\Lambda)},$$

$k = 1, 2, \dots,$ with some implicit constant independent of the kernel T, weights a, b and the index k. In other words, $T_{ba} \in \mathbf{S}_{q,\infty}$ with

$$\frac{1}{q} = \frac{1}{2} + \frac{l}{d},$$

and

$$\|T_{ba}\|_{q,\infty} \lesssim \left[\int \|T(t,\ \cdot\ )\|^2_{\mathsf{H}^l} |b(t)|^2\,dt\right]^{\frac{1}{2}} \|a\|_{\mathsf{L}^2(\Lambda)}.$$

Operators with a homogeneous kernel: [BS1977], [BS1970].

Proposition. Let $X, Y \subset \mathbb{R}^d, d \geq 1$, be bounded Borel sets. Let $T : \mathsf{L}^2(Y) \to \mathsf{L}^2(X)$ be the operator with the kernel

$$T(x, y) = \rho_1(x)|x - y|^\alpha \phi(x, y)\rho_2(y),$$

where $\alpha > -d$, $\rho_1 \in \mathsf{L}^\infty(X), \rho_2 \in \mathsf{L}^\infty(Y)$, and $\phi \in \mathsf{C}^\infty(\overline{X} \times \overline{Y})$.

Operators with a homogeneous kernel: [BS1977], [BS1970].

Proposition. Let $X, Y \subset \mathbb{R}^d, d \geq 1$, be bounded Borel sets. Let $T : \mathsf{L}^2(Y) \to \mathsf{L}^2(X)$ be the operator with the kernel

$$T(x, y) = \rho_1(x)|x - y|^\alpha \phi(x, y)\rho_2(y),$$

where $\alpha > -d$, $\rho_1 \in \mathsf{L}^\infty(X), \rho_2 \in \mathsf{L}^\infty(Y)$, and $\phi \in \mathsf{C}^\infty(\overline{X} \times \overline{Y})$. Then for $p^{-1} = 1 + \alpha d^{-1}$ we have

$$\mathsf{g}_p(T) = \mathsf{G}_p(T) = \mu_{\alpha,d} \int\limits_{X \cap Y} |\rho_1(x)\phi(x, x)\rho_2(x)|^p dx,$$

with

$$\mu_{\alpha,d} = \frac{1}{\Gamma\big(d/2 + 1\big)} \left[\frac{\Gamma\big((d + \alpha)/2\big)}{\pi^{\alpha/2}|\Gamma(-\alpha/2)|}\right]^p, \qquad \alpha \neq 0, 2, 4, \dots,$$

$$\mu_{\alpha,d} = 0, \quad \alpha = 0, 2, 4, \dots.$$

Bibliography

1. AJPR2002 A. B. Aleksandrov, S. Janson, V.V. Peller and R. Rochberg, *An interesting class of operators with unusual Schatten-von Neumann behavior*. *Function spaces, interpolation theory and related topics (Lund, 2000)*, 61–149, de Gruyter, Berlin, 2002.
2. BS1970 M. S. Birman and M. Z. Solomyak, *Asymptotics of the spectrum of weakly polar integral operators*. Izv. Akad. Nauk SSSR Ser. Mat. **34**: 1142–1158, 1970.
3. BS1977 M. S. Birman and M. Z. Solomyak, *Estimates for the singular numbers of integral operators. (Russian)*. Uspehi Mat. Nauk **32(1(193))**: 17–84, 1977.
4. BS M. S. Birman and M. Z. Solomyak, *Spectral Theory of Selfadjoint Operators in Hilbert Space*. Mathematics and its Applications (Soviet Series), D. Reidel, 1987. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller.
5. Sob2020 A. V. Sobolev, *Eigenvalue estimates for the one-particle density matrix*. Arxiv, 2008.10935.