

Toepplitz Determinants

and

Painlevé Transcendents.

( A Riemann-Hilbert point of view )

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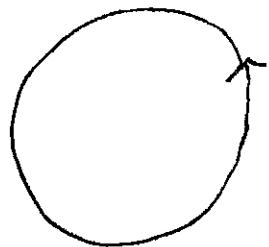
# 1. Toeplitz Determinants

(1)

Definition.

$$\varphi(z) \in L_1(\mathbb{C})$$

$$C : |z|=1, z = e^{i\theta}, 0 \leq \theta < 2\pi$$



$$\varphi_k := \int_C \varphi(z) z^{-k} \frac{dz}{2\pi iz} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \varphi(e^{i\theta}) d\theta$$

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$$T_n[\varphi] = \{ \varphi_{j-k} \}_{j,k=0,\dots,n-1}$$

=

$$\begin{pmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \dots & \varphi_{-n+1} \\ \varphi_1 & \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n+2} \\ \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & \varphi_{-1} & \\ \varphi_{n-1} & \dots & \dots & \dots & \varphi_1 & \varphi_0 \end{pmatrix}$$

$$D_n[\varphi] = \det T_n[\varphi]$$

Otto Toeplitz, 1907

(Habilitationsschrift)

## Applications:

- Statistical mechanics, Ising model, exactly solvable quantum mechanical models ( spin chains  $XX0$ ,  $XI$  )
- Orthogonal polynomials, spectral theory of difference operators
- Random matrices, random permutations, growth processes, random Young tableaux fillings, packing.

Main questions of the theory

$$D_n[\varphi] \approx ?$$

$$n \rightarrow \infty$$

$$\varphi = \varphi(z; \vec{t})$$

$$D_n[\varphi] \approx ?$$

$$n \rightarrow \infty, t_H \rightarrow t_0$$

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## Classic: Strong Szegő Theorem.

- $\varphi(z) \neq 0$
- $\text{Ind } \varphi(z) = 0$
- $\sum_{k=-\infty}^{\infty} |k| |(\log \varphi)_k|^2 < \infty$

$$(\log \varphi)_k = \frac{1}{2\pi i} \int_C \log \varphi(z) z^{-k} \frac{dz}{z}$$

Then

~~D~~

$$D_n[\varphi] = e^{n(\log \varphi)_0} e^{E[\varphi]} (1 + o(1)), n \rightarrow \infty$$

$$E[\varphi] = \sum_{k=1}^{\infty} k (\log \varphi)_k (\log \varphi)_{-k}$$

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## 2. Tzepitz determinants and OPUC.

$$\varphi(z) \longmapsto \left\{ P_k(z) \right\}_{k=1}^{\infty}, \quad \left\{ \hat{P}_k(z) \right\}_{k=1}^{\infty};$$

$$P_k(z) = z^k + \dots, \quad \int_C z^{-j} P_k(z) \varphi(z) \frac{dz}{2\pi i z} = h_k \delta_{kj}$$

j = 0, \dots, K

$$\hat{P}_k(z) = z^k + \dots, \quad \int_C z^j \hat{P}_k(z^{-1}) \varphi(z) \frac{dz}{2\pi i z} = h_k \delta_{kj}$$

j = 0, \dots, K

Remark 1.

$$\int_C P_k(z) \hat{P}_j(z^{-1}) \varphi(z) \frac{dz}{2\pi i z} = h_k \delta_{kj}$$

Remark 2.

if  $\varphi(z) = \overline{\varphi(z)}$ , then

$$\hat{P}_k(z) = \overline{\hat{P}_k(\bar{z})}$$

and

$$\left\{ \begin{array}{l} \hat{P}_k(z|\varphi) \\ \parallel \\ \overline{\hat{P}_k(\bar{z}|\bar{\varphi})} \end{array} \right.$$

$$\int_C P_k(z) \overline{P_j(z)} \varphi(z) \frac{dz}{2\pi i z} = h_k \delta_{kj}$$

Remark 3.

$P_n(z)$ -exists if  $D_n[\varphi] \neq 0$

if  $D_{n+1}[\varphi] D_n[\varphi] \neq 0$

then

$$\boxed{\frac{D_{n+1}}{D_n} = h_n}$$

$$D_n = \prod_{j=0}^{n-1} h_j$$

Remark 4.

$$\text{Define: } Q_k(z) := - \frac{1}{h_k} z^k \hat{P}_k(z^{-1})$$

$$\varphi_\gamma(z) := \gamma \varphi(z) + (1-\gamma)$$

$$P_k(z), \hat{P}_k(z) \equiv P_k(z|\gamma), \hat{P}_k(z|\gamma)$$

Then:

$$\ln D_n[\varphi]$$

$$= - \int_0^1 \left[ \int_C \left( P'_n(z) Q_{n-1}(z) - P_n(z) Q'_{n-1}(z) \right) z^{-n} \frac{\varphi(z)-1}{2\pi i} dz \right] d\gamma$$

### 3. The Riemann-Hilbert method

$$\varphi(z) \mapsto \{P_k(z)\}, \{\hat{P}_k(z)\}$$



$$\Psi(z) := \begin{cases} P_n(z) & \int_C \frac{P_n(s)s^{-n}\varphi(s)}{s-z} \frac{ds}{2\pi i} \\ Q_{n-1}(z) & \int_C \frac{Q_{n-1}(s)s^{-n}\varphi(s)}{s-z} \frac{ds}{2\pi i} \end{cases}$$

$$\left\{ Q_k(z) := -\frac{1}{h_k} z^k \hat{P}_k(z^{-1}) \right\}$$

(Baik, Deift, Johansson)

Claim 1:

- (a)  $\mathbb{Y}(z) \in H(\mathbb{C} \setminus C)$

- (b)  $\mathbb{Y}_+(z) = \mathbb{Y}_-(z) \begin{pmatrix} 1 & z^{-n}\varphi(z) \\ 0 & 1 \end{pmatrix}$

$$z \in \mathbb{C}, \quad \mathbb{Y}_\pm(z) = \lim_{\tilde{z} \rightarrow \pm \text{side of } C} \mathbb{Y}(\tilde{z})$$

- (c)  $\mathbb{Y}(z) = (I + O(z^{-1})) z^{n\beta_3}, \quad z \rightarrow \infty$

$$\beta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z^{n\beta_3} = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

Proof:

(a) - obvious

(b) - direct corollary of Plemelj-Sokhotsky formula:

$$F(z) := \int_C \frac{f(s)}{s-z} \frac{ds}{2\pi i}$$

$$F_+(z) - F_-(z) = f(z).$$

(c) - follows from orthogonality of  $P_n(z)$ ,  
 $P_{n-1}(z)$ .

Indeed we have:

$$\mathbb{Y}_{11}(z) = z^n + O(z^{n-1})$$

$$\mathbb{Y}_{12}(z) = - \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \int_C s^{\ell-n} P_n(s) \varphi(s) \frac{ds}{2\pi i}$$

$$= - \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \int_C s^{\ell-n+1} P_n(s) \varphi(s) \frac{ds}{2\pi i s} \\ = O\left(\frac{1}{z^{n+1}}\right)$$

$$\mathbb{Y}_{21}(z) = O(z^{n-1})$$

$$\mathbb{Y}_{22}(z) = - \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \int_C s^{\ell-n+1} Q_{n-1}(s) \varphi(s) \frac{ds}{2\pi i s}$$

$$= \frac{1}{h_{n-1}} \sum_{\ell=0}^{\infty} \frac{1}{z^{\ell+1}} \int_C s^\ell \hat{P}_{n-1}(s^{-1}) \varphi(s) \frac{ds}{2\pi i s}$$

$$\left( Q_{n-1}(s) = -\frac{1}{h_{n-1}} s^{n-1} \hat{P}_{n-1}(s^{-1}) \right)$$

$$= \frac{1}{z^n} + O\left(\frac{1}{z^{n+1}}\right)$$

Hence,

$$Y(z) = \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 + O(z) & O(z^{-1}) \\ O(z^{-1}) & 1 + O(z^{-1}) \end{pmatrix} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

$$= \left( I + O(z^{-1}) \right) z^{n b_3} \quad Q.E.D.$$

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Claim 2:

(a), (b), (c) - determine  $\tilde{Y}(z)$  uniquely

Indeed, suppose  $\tilde{Y}(z)$  is another such function.  
and consider

$$X(z) := \tilde{Y}(z) Y^{-1}(z).$$

Observe that

$\det Y(z) \in H(\mathbb{C} \setminus C)$ ,  $\det Y_+(z) = \det Y_-(z)$ ,

$$\det Y(z) \rightarrow 1, z \rightarrow \infty$$

by Liouville theorem,  $\det Y(z) = 1$  and we

have that

$$X(z) \in H(C \setminus C),$$

$$(1) \quad X_+(z) = \underbrace{Y_+(z)}_{\sim} Y_+^{-1}(z) = \underbrace{Y_-(z)}_{\sim} G(z) \left( Y_-(z) G(z) \right)^{-1}$$

$$(2) \quad G(z) := \begin{pmatrix} 1 & z^{-n} \varphi(z) \\ 0 & 1 \end{pmatrix}$$

$$= \underbrace{Y_-(z)}_{\sim} Y_-^{-1}(z) = \underbrace{X_-(z)}_{\sim}$$

and, finally, as  $z \rightarrow \infty$ ,

$$X(z) = (I + O(z^{-1})) z^{n\beta_3} \left[ (I + O(z^{-1})) z^{n\beta_3} \right]^{-1}$$

$$(3) \quad = I + O(z^{-1})$$

$$(1) - (3) \Rightarrow \boxed{X(z) = I.}$$

Observe that we also have that

$$\frac{D_{n+1}}{D_n} = h_n = Y_{12}(0) \quad \text{and}$$

$$\ln D_n[\varphi] = - \int_0^L \left\{ \left( Y_{11}'(z) Y_{21}(z) - Y_{11}(z) Y_{21}'(z) \right) \right.$$

$$\left. \times z^{-h} \frac{\varphi(z) - 1}{2\pi i} dz \right\} d\varphi \quad (22.1)$$

where  $Y(z) = Y(z | \varphi_\delta)$

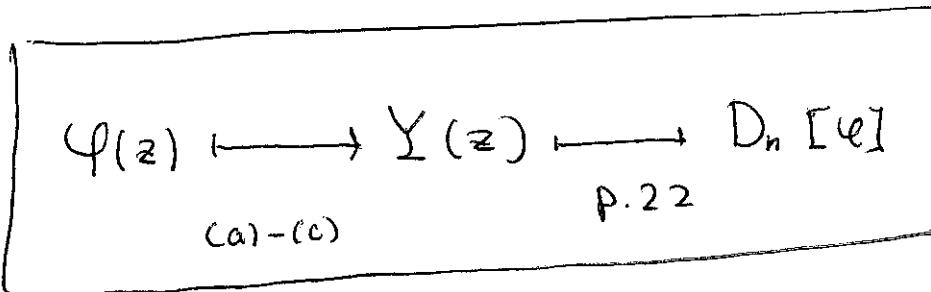
$$\varphi_\delta(z) = \gamma \varphi(z) + 1 - \gamma$$



## Key methodological idea:

"Forget" about orthogonal polynomials, use (a) - (c)

To define  $Y(z)$  by given  $\Psi(z)$  and formulae  
of p.22 to determine  $D_n[\Psi]$ :

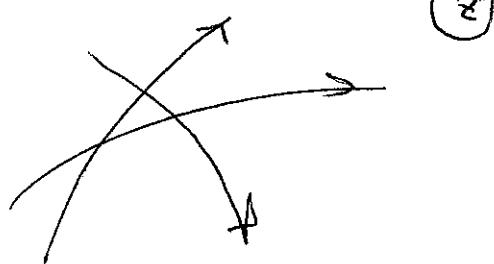


In other words, the asymptotic analysis  
of  $D_n[\Psi]$  is reduced to the  
asymptotic solution of the

Riemann-Hilbert Problem (a) - (c)

In general, by a RH problem we shall understand the following problem of complex analysis:

Given: oriented contour  $\Gamma$ :



and  $G_e(z)$ :  $G_e: \Gamma \rightarrow \text{GL}(N, \mathbb{C})$

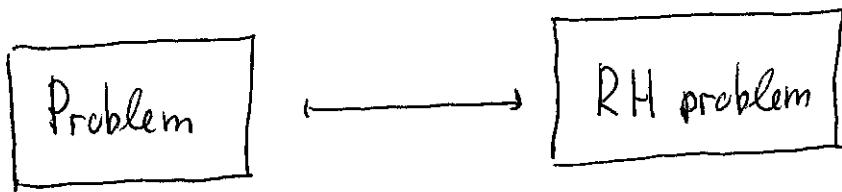
Find: The  $N \times N$   $Y(z)$  such that:

- .  $Y(z) \in H(\mathbb{C} \setminus \Gamma)$

- .  $Y_+(z) = Y_-(z) G_e(z), z \in \Gamma$

- .  $Y(z) \rightarrow I, z \rightarrow \infty$

## The Riemann-Hilbert method:



$[G(z), G(z')] = 0 \Rightarrow$  RH can be solved

explicitly:

$$Y(z) = \exp \left\{ \frac{1}{2\pi i} \int_{\Gamma} \frac{\ln G(s)}{s - z} ds \right\}$$

if  $[G(z), G(z')] \neq 0$  — no explicit formula,

but can be used for asymptotic analysis  
as effectively as contour integral.

RH = non-abelian contour integral  
Classical steepest descent method  $\rightarrow$   
nonlinear steepest descent method  
of Deift & Zhou.

#### 4. Strong Szegő Theorem

by RHM.

Basic idea:

$$Y(z) \rightarrow T(z) \rightarrow S(z) \rightarrow R(z)$$

Exact transformations

$$G_R(z) = I + o(1)$$

$\Downarrow$

$$R(z) = I + o(1)$$

Now, we are going to apply this scheme  
to the RH case.

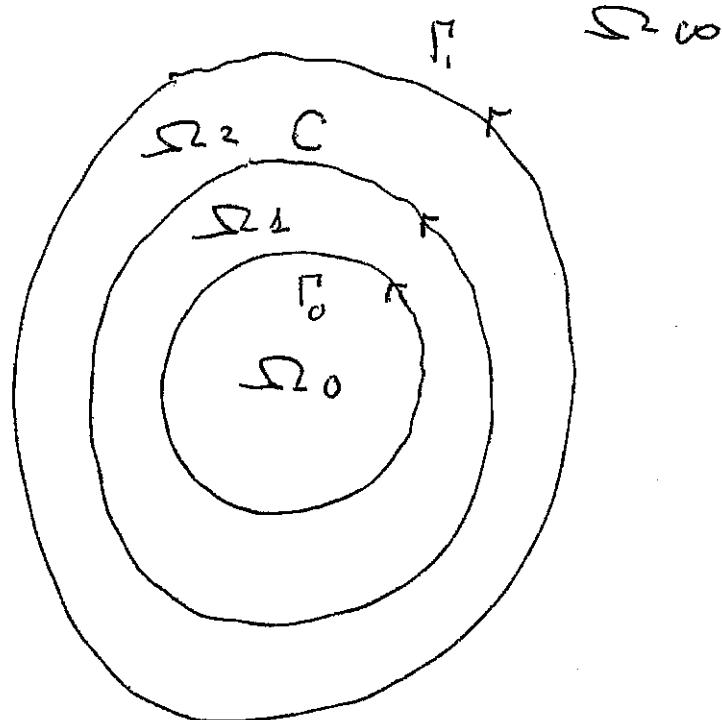
## 4.1 Asymptotics of $\underline{Y}(z)$

$$\underline{Y}(z) \mapsto \overline{T}(z) = \begin{cases} \underline{Y}(z) z^{-n\beta_3} & |z| > 1 \\ \underline{Y}(z) & |z| < 1 \end{cases}$$

- $\overline{T}_+(z) = \overline{T}_-(z) \begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} \quad z \in C$

- $\overline{T}(\infty) = \overline{I}$

$$\begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^n\varphi^{-1} & 1 \end{pmatrix}$$



$$T(z) \mapsto S(z) = T(z) \left\{ \begin{array}{l} \left( \begin{smallmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{smallmatrix} \right) \text{ in } \Omega_2 \\ \left( \begin{smallmatrix} 1 & 0 \\ -z^n \varphi^{-1} & 1 \end{smallmatrix} \right) \text{ in } \Omega_1 \\ I \quad \text{in } \Omega_0 \cup \Sigma_\infty \end{array} \right.$$

! Key assumption:

$\varphi(z)$  - analytic in an

annulus around  $C$

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$$S_+(z) = S_-(z) G_S(z)$$

$$z \in \Gamma_S = \Gamma_0 \cup C \cup \Gamma_1$$

$G_S(z) :$

$$\begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix} \stackrel{||}{=} I + O(e^{-\epsilon n})$$

$$\begin{pmatrix} 0 & \varphi(z) \\ -\varphi^{-1}(z) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix} = I + O(e^{-\epsilon n})$$

$$S(\infty) = I$$

Heuristically,

$$\boxed{S(z) \approx P^{(\infty)}(z)}$$

"global" parametrix

- $P^{(\infty)}(z) \in H(\mathbb{C} \setminus C)$

- $P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & \varphi(z) \\ -\varphi'(z) & 0 \end{pmatrix}, z \in C$

- $P^{(\infty)}(\infty) = I$

Rigorous statement:

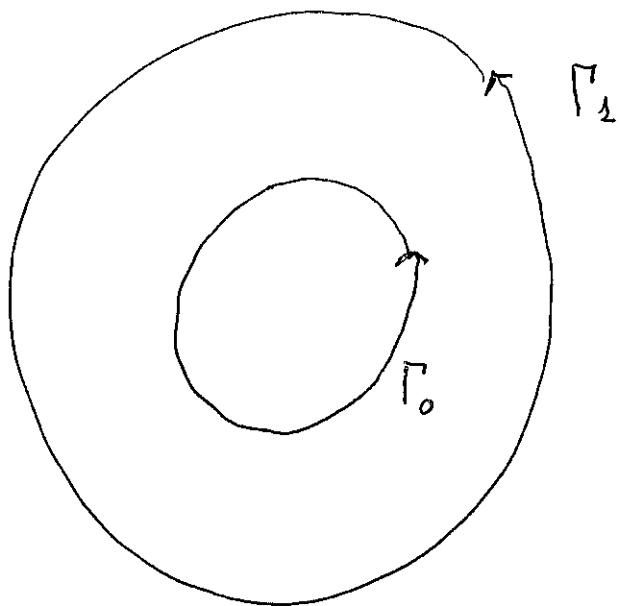
$$\boxed{S(z) = \left( I + O\left(\frac{e^{-\epsilon n}}{1+|z|}\right) \right) P^{(\infty)}(z)}$$

$n \rightarrow \infty$ , uniformly on  $z$

A little bit more details:

$$S(z) \mapsto R(z) = S(z) [P^{(n)}_{(z)}]^{-1}$$

$\Gamma_R, G_R$ :



$$G_R = P^{(n)}(z) G_S(z) [P^{(n)}(z)]^{-1}$$

$$= J + O(e^{-\epsilon n})$$

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Observe:

$$\begin{pmatrix} 0 & \varphi(z) \\ -\varphi'(z) & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{J}_-(z) & 0 \\ 0 & \mathcal{J}_-(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{J}_+(z) & 0 \\ 0 & \mathcal{J}_+(z) \end{pmatrix}$$

$$\mathcal{J}(z) = \exp \left[ \frac{1}{2\pi i} \int_C \frac{\log \varphi(\xi)}{\xi - z} d\xi \right]$$

$$\left( \frac{\mathcal{J}_+(z)}{\mathcal{J}_-(z)} = \varphi(z) ! \right)$$

↳ Szegö Funktion



$$P^{(\infty)}(z) = \begin{cases} \begin{pmatrix} \mathcal{J}(z) & 0 \\ 0 & \mathcal{J}'(z) \end{pmatrix} & |z| > 1 \\ \begin{pmatrix} 0 & \mathcal{J}(z) \\ -\mathcal{J}'(z) & 0 \end{pmatrix} & |z| < 1 \end{cases}$$

asymptotics for  $Y(z; n)$  follows.

## 4.2 Asymptotics of the determinant

We are going to use identity (22.17):

$$\ln D_n = - \int_0^A \int_C \left( Y_{11}^* Y_{21} - Y_{21}^* Y_{11} \right) z^n \frac{\varphi(z)-1}{2\pi i} dz dy$$

$$Y \equiv Y(z|\varphi_f)$$

$$\varphi_f(z) = \gamma \varphi(z) + 1 - \gamma$$

From the previous analysis we have,

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$$Y_+(z) = T_+(z) = S_+(z) \begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix}$$

$$= (I + O(e^{-\varepsilon n})) P_+^{(\omega)}(z) \begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix}$$

$$= (I + O(e^{-\varepsilon n})) \begin{pmatrix} 0 & J_+(z) \\ -J_+^{-1}(z) & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix}$$

Hence:

$$Y_{11}(z) = \mathcal{L}_+(z) z^n \varphi_j^{-1}(z) + O(e^{-\epsilon n})$$

$$Y_{21}(z) = - \mathcal{L}_+^{-1}(z) + O(e^{-\epsilon n})$$

$n \rightarrow \infty$

and :

$$Y_{11}^{-1} = \mathcal{L}_+^{-1} z^n \varphi_j^{-1} + n z^{n-1} \mathcal{L}_+(z) \varphi_j^{-1}(z)$$

$$- \mathcal{L}_+(z) z^n \varphi_j^{-2} \varphi_j^{-1}$$

$$Y_{21}^{-1} = \mathcal{L}_+^{-2}(z) \mathcal{L}_+^{-1}(z)$$

Therefore:

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$$- \left( Y'_{11} Y_{21} - Y_{21} Y_{11} \right) z^{-n} \frac{\varphi^{-1}}{2\pi i}$$

$$= 2 d_+^{-1} d_+^1 \varphi_2^{-1} \frac{\varphi^{-1}}{2\pi i} + \frac{n}{z} \varphi_2^{-1} \frac{\varphi^{-1}}{2\pi i} - \varphi_2^{-2} \varphi_2^1 \frac{\varphi^{-1}}{2\pi i} \\ + O(e^{-en})$$

and hence,

$$\ln D_n = I_1 + I_2 + I_3 + O(e^{-en})$$

$$I_1 = \int_0^1 \int_C \frac{n}{z} \varphi_2^{-1} \frac{\varphi^{-1}}{2\pi i} dz dt$$

$$I_2 = - \int_0^1 \int_C \varphi_2^{-2} \varphi_2^1 \frac{\varphi^{-1}}{2\pi i} dz dt$$

$$I_3 = 2 \int_0^1 \int_C d_+^{-1} d_+^1 \varphi_2^{-1} \frac{\varphi^{-1}}{2\pi i} dz dt$$

$I_1:$

Observe that

$$\varphi - 1 = \frac{d\varphi}{d\gamma}, \quad \text{hence:}$$

$$I_1 = \frac{1}{2\pi i} \int_0^1 \int_C \frac{n}{z} \varphi_j^{-1} \frac{d\varphi}{d\gamma} dz d\gamma$$

$$= \frac{1}{2\pi i} \int_0^1 \int_C \frac{n}{z} \frac{d}{d\gamma} \ln \varphi_j(z) dz d\gamma$$

$$= \frac{1}{2\pi i} \int_C \frac{n}{z} \ln \varphi(z) dz = \boxed{n (\ln \varphi)_0}$$

$I_2:$

Observe that

$$\left\{ \begin{array}{l} \varphi_\delta = \varphi + 1 - \delta \\ \end{array} \right.$$

$$\varphi_\delta^{-2} \varphi'_\delta (\varphi - 1) = \varphi_\delta^{-2} \varphi'_\delta \frac{1}{\delta} (\varphi_\delta - 1)$$

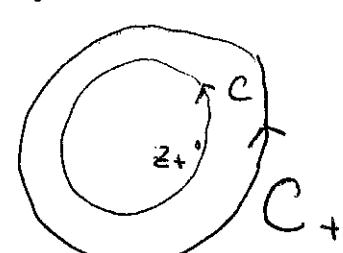
$$= \frac{1}{\delta} \frac{\varphi'_\delta}{\varphi_\delta} - \frac{1}{\delta} \frac{\varphi'_\delta}{\varphi_\delta^2} \quad \text{and hence}$$

$$I_2 = -\frac{1}{2\pi i} \iint_C \left( \frac{1}{\delta} \frac{\varphi'_\delta}{\varphi_\delta} - \frac{1}{\delta} \frac{\varphi'_\delta}{\varphi_\delta^2} \right) dz dr$$

$$= -\frac{1}{2\pi i} \int_0^1 \left[ \frac{1}{\delta} \ln \varphi_\delta \Big|_C + \frac{1}{\delta} \varphi_\delta^{-1} \Big|_C \right] d\delta = \boxed{0}$$

$\mathcal{I}_3 :$

Observe that

$$\ln d_+(z) = \frac{1}{2\pi i} \int_C \frac{\ln \varphi_s(s)}{s - z_+} ds$$


$$= \frac{1}{2\pi i} \int_{C_+} \frac{\ln \varphi_s(s)}{s - z} ds$$

$C_+$

and

$$J_+^{-1} J_+^1 = \frac{1}{2\pi i} \int_{C_+} \frac{\ln \varphi_s(s)}{(s - z)^2} ds$$

we have then,

I<sub>3</sub>

$$= 2 \iint_{\substack{0 \\ C \\ C_+}}^1 \frac{\ln \varphi_r(s)}{(s-z)^2} \frac{d}{ds} \ln \varphi_r(z) \frac{ds dz dr}{(2\pi i)^2}$$

$$= \iint_{\substack{0 \\ C \\ C_+}}^1 \frac{1}{(s-z)^2} \frac{d}{ds} (\ln \varphi_r(z) \ln \varphi_r(s)) \frac{ds dz dr}{(2\pi i)^2}$$

$$= \iint_{\substack{C \\ C_+}} \frac{\ln \varphi(z) \ln \varphi(s)}{(s-z)^2} \frac{ds dz}{(2\pi i)^2}$$

$$= \int\limits_C \ln \varphi(z) \left( \frac{d}{dz} \int\limits_{C_+} \frac{\ln \varphi(s)}{s-z} \frac{ds}{2\pi i} \right) \frac{dz}{2\pi i}$$

$$= \int\limits_C \ln \varphi(z) \left( \frac{d}{dz} \sum_{k=0}^{\infty} z^k \int\limits_{C_+} \ln \varphi(s) s^{-k-1} \frac{ds}{2\pi i} \right) \frac{dz}{2\pi i}$$

$$= \int\limits_C \ln \varphi(z) \left( \frac{d}{dz} \sum_{k=0}^{\infty} z^k (\ln \varphi)_k \right) \frac{dz}{2\pi i}$$

$$= \sum_{k=0}^{\infty} k (\ln \varphi)_k \int\limits_C z^{k-1} \ln \varphi(z) \frac{dz}{2\pi i}$$

$$= \boxed{ \sum_{k=0}^{\infty} k (\ln \varphi)_k (\ln \varphi)_{-k} }$$

Assembling all three integrals,

$$\ln D_n = n (\ln \varphi)_0 + \sum_{k=0}^{\infty} k (\ln \varphi)_{ik} (\ln \varphi)_{-is} + O(e^{-kn})$$

$$n \rightarrow \infty$$

SST.

(Unsager, Kaufman, Szegő, Widom,

Ibragimov, Gelchinsky

Johansson

## 5. Singular weights.

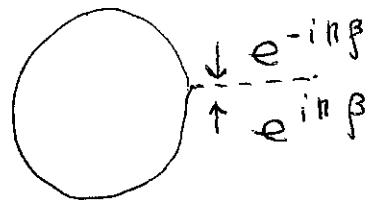
### The Fisher-Hartwig asymptotics.

$$\varphi(z) = e^{V(z)} |z-1|^{2d} z^\beta e^{-i\pi\beta}$$

$$2d > -1 \quad 0 < \arg z < 2\pi.$$

$g \in \mathbb{C}$   $V(z)$  - anal. in annulus

Fact singularity  $\oplus$  jump singularity



$$V(z) = \sum_{k=-\infty}^{\infty} z^k V_k$$

The answer:

Fisher-Hartwig Conjecture

Widom

Basor

Böttcher & Silbermann

Ehrhardt

$$\ln D_n = n V_0 + (\alpha^2 - \beta^2) \ln n + \sum_{k=1}^{\infty} k V_k V_{-k}$$

$$- (\alpha - \beta) \sum_{k=1}^{\infty} V_k - (\alpha + \beta) \sum_{k=1}^{\infty} V_{-k}$$

$$+ \ln \frac{G_e(1+\alpha + \beta) G_e(1+\alpha - \beta)}{G_e(1+2\alpha)} + o(1)$$

$$n \rightarrow \infty$$

$G_e(x)$  - Barnes'  $G_e$ - function

$$G_e(x+1) = \Gamma(x) G_e(x), \quad G_e(1) = 1, \quad G_e(-k) = 0$$

$$k = 0, 1, \dots$$

$$G_e(x) = (2\pi)^{\frac{x}{2}} e^{-\frac{x(x+1)}{2}} \gamma_E \frac{x^x}{2^x}$$

$$\times \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^n e^{-x + \frac{x^2}{2n}}$$

## Riemann-Hilbert analysis

$$\left( \begin{array}{c} Y \rightarrow T \rightarrow S \rightarrow R \end{array} \right)$$

Important remark:

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} \varphi(z) \\ 0 & 1 \end{pmatrix}, \quad z \in C \setminus \{z\}$$

- at  $z=1$ ,  $Y(z)$  behaves as follows

$$Y(z) = \begin{pmatrix} O(1) & O(1) + O(|z-1|^{2\delta}) \\ O(1) & O(1) + O(|z-1|^{2\delta}) \end{pmatrix}$$

Now, as before,

$$Y \mapsto T(z) = Y(z) \left\{ \begin{array}{l} z^{-n} b_3, \quad |z| > 1 \\ I, \quad |z| < 1. \end{array} \right.$$

- $T_+(z) = T_-(z) \begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} \quad z \in C \setminus \{1\}$
- $T(\infty) = I \quad , \quad \text{at } z=1 - \text{as } Y(z)$

Once again,

$$\begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^n\varphi^{-1} & 1 \end{pmatrix}$$

But !

$$\varphi(z) = e^{V(z)} (z-1)^{2\alpha} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)}$$

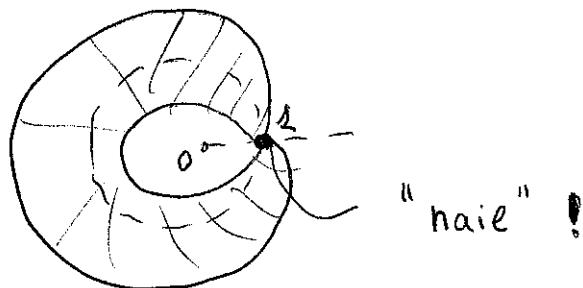
$$\left| z-1 \right|^{2\alpha} = \left( (z-1)(\bar{z}-1) \right)^\alpha = \left( (z-1)(1-z)\frac{1}{z} \right)^\alpha = (z-1)^{2\alpha} z^{-\alpha} e^{-i\pi\alpha}$$

$$z^{-\alpha+\beta} : \quad \overbrace{\quad}^{\text{---}} \quad 0 < \arg z < 2\pi$$

$$(z-1)^{2\alpha} : \quad \overbrace{\quad}^{\text{---}} \quad 0 < \arg(z-1) < 2\pi$$

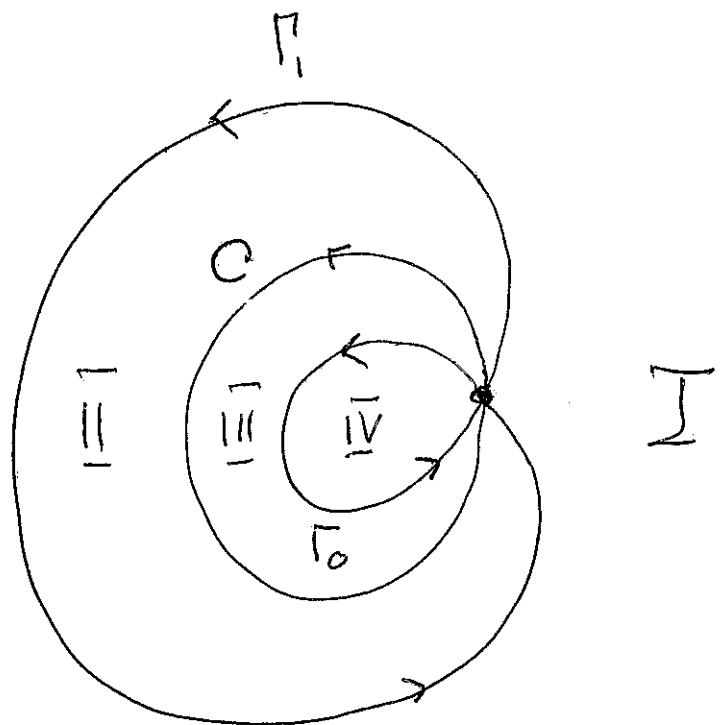
↓

analiticity in



5.1  $T \rightarrow S$  (Lerzer opening)

(46)



$$S(z) = T(z) \begin{cases} I \text{ in } I, \bar{IV} \\ \left( \begin{array}{cc} 1 & 0 \\ z^n \varphi & 1 \end{array} \right) \text{ in } II \\ \left( \begin{array}{cc} 1 & 0 \\ -z^n \varphi & 1 \end{array} \right) \text{ in } III \end{cases}$$

S - problem:

$$\Gamma_S = \Gamma_0 \cup C \cup \Gamma_1$$

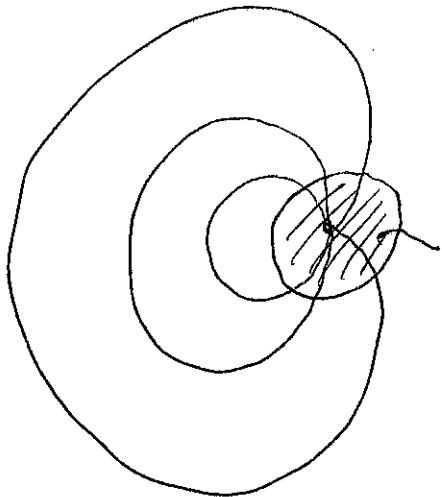
$$G_S = \left\{ \begin{array}{ll} \begin{pmatrix} 1 & 0 \\ z^n \varphi^{-1} & 1 \end{pmatrix} & \text{on } \Gamma_0 \\ \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} & \text{on } C \\ \begin{pmatrix} 1 & 0 \\ z^{-n} \varphi^{-1} & 1 \end{pmatrix} & \text{on } \Gamma_1 \end{array} \right.$$

at  $z = 1$ :

$$S(z) = \begin{pmatrix} O(1) & O(1) + O(|z-1|^{2d}) \\ O(1) & O(1) + O(|z-1|^{2d}) \end{pmatrix} \text{ in } I, IV,$$

$$S(z) = \begin{pmatrix} O(1) + O(|z-1|^{-2d}) & O(1) + O(|z-1|^{2d}) \\ O(1) + O(|z-1|^{-2d}) & O(1) + O(|z-1|^{2d}) \end{pmatrix} \text{ in } II, III$$

Observe:



outside this  $U$ -neighborhood of  $z=1$ ,

$$G_{\text{es}} \Big|_{\Gamma_0, \Gamma_1} = I + O(e^{-\epsilon n})$$

Hence, heuristically,

$$S(z) \approx P^{(\infty)}(z) \quad \text{in } \mathbb{C} \setminus U$$

global parametrix

$$S(z) \approx P^{(1)}(z) \quad \text{in } U$$

local parametrix.

## 5.2 Global parametrix

- $P^{(\infty)}(z) \in H(C \setminus C)$

- $P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & \varphi(z) \\ -\varphi^{-1}(z) & 0 \end{pmatrix}$   
 $z \in C \setminus \{1\}$

- $P^{(0)}(w) = I$

"Siegels" function

$$\mathcal{D}_+(z) = \mathcal{D}_-(z) \varphi(z)$$

$$\varphi(z) = e^{V(z)} (z-1)^{\frac{d}{2}} z^{\frac{d}{2} + \beta} e^{-i\pi(d+\beta)}$$

$$= e^{V_+(z) + V_-(z)} (z-1)^{d+\beta} e^{-i\pi(d+\beta)} \left(\frac{z-1}{z}\right)^{d-\beta}$$

$$V(z) = \sum_{k=-\infty}^{\infty} V_k z^k$$

$$V_+(z) = \sum_{k=0}^{\infty} V_k z^k, \quad V_-(z) = \sum_{k=-\infty}^{-1} z^k V_k$$

↓

$$\mathcal{D}(z) = \begin{cases} e^{V_+} (z-1)^{\alpha+\beta} e^{-i\pi(\alpha+\beta)}, & |z| < 1, \\ e^{-V_-(z)} \left(\frac{z-1}{z}\right)^{\beta-\alpha}, & |z| > 1 \end{cases}$$

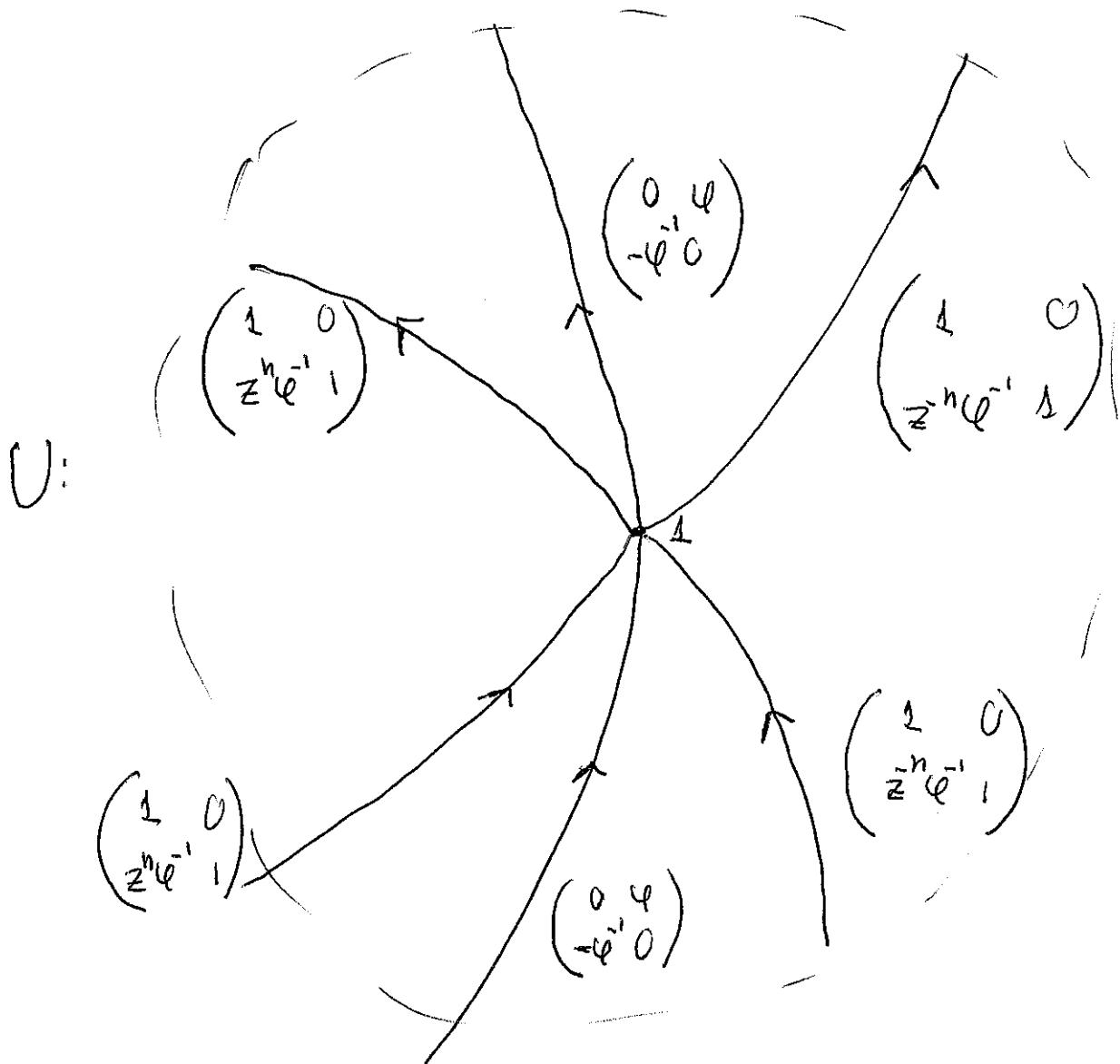
and

$$P^{(0)}(z) = \begin{cases} \begin{pmatrix} \mathcal{D}(z) & 0 \\ 0 & \mathcal{D}^{-1}(z) \end{pmatrix} & |z| > 1 \\ \begin{pmatrix} 0 & \mathcal{D}(z) \\ -\mathcal{D}^{-1}(z) & 0 \end{pmatrix} & |z| < 1 \end{cases}$$

### 5.3 Local parametrix

51

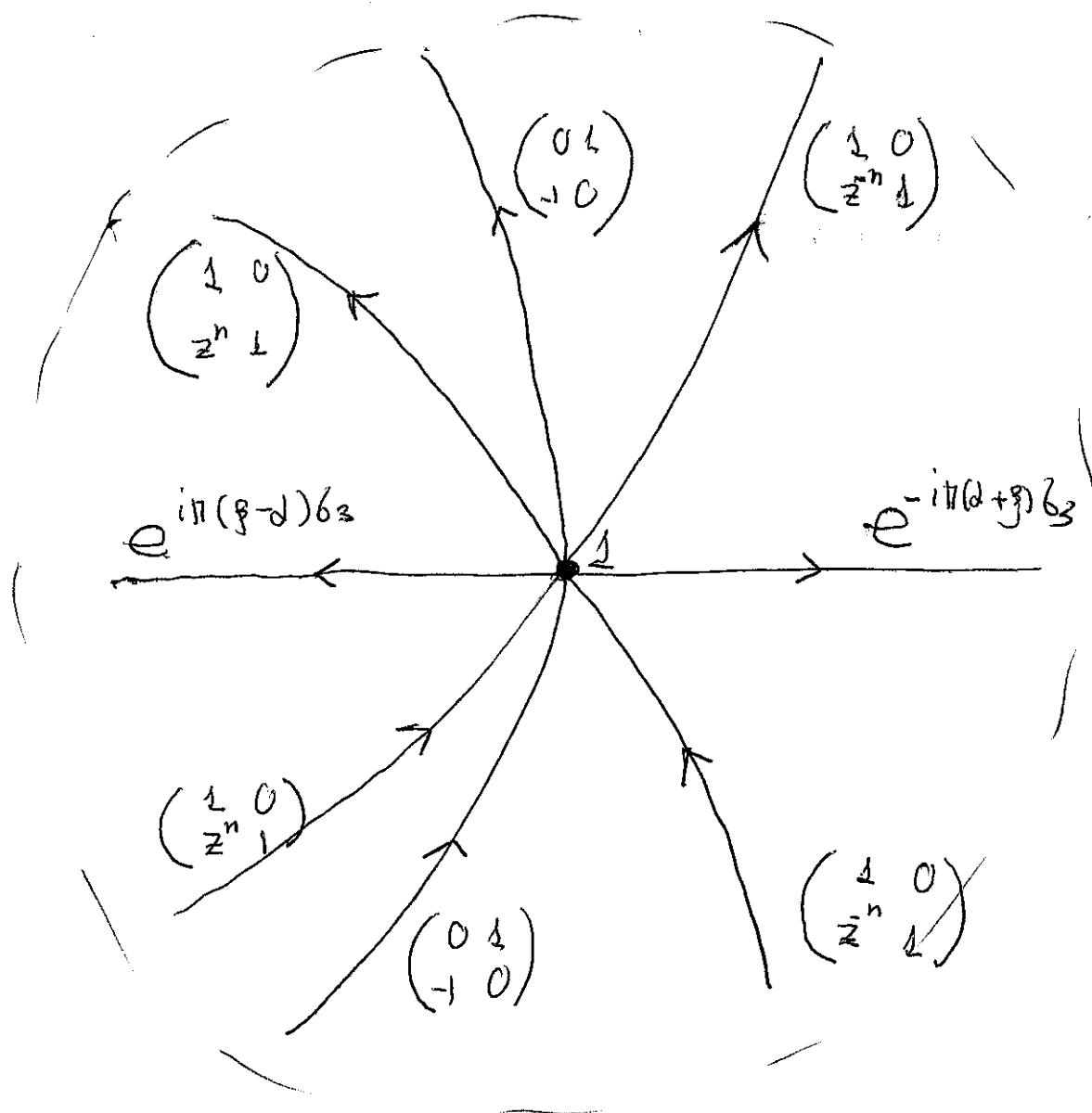
Locally, for  $S(z)$ , we have:



$$S \mapsto \tilde{S} = S \varphi^{\delta_{3/2}} \quad (\text{in } U!)$$

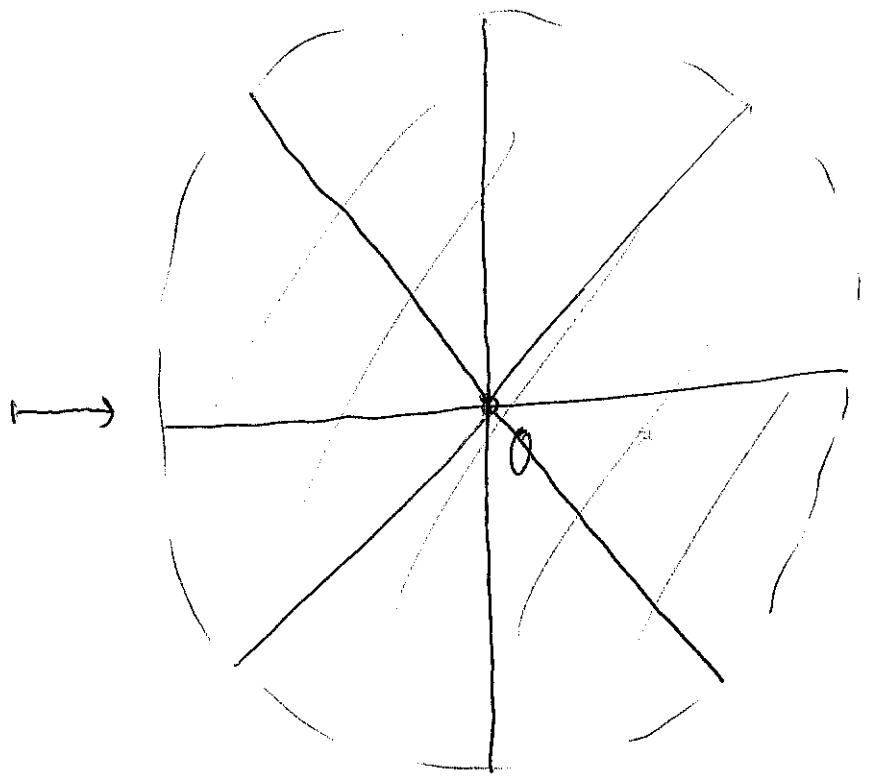
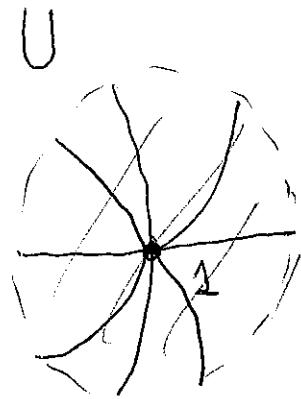
(52)

$\tilde{S}$ -RH problem:



$$z \mapsto \xi := n \ln z$$

(53)

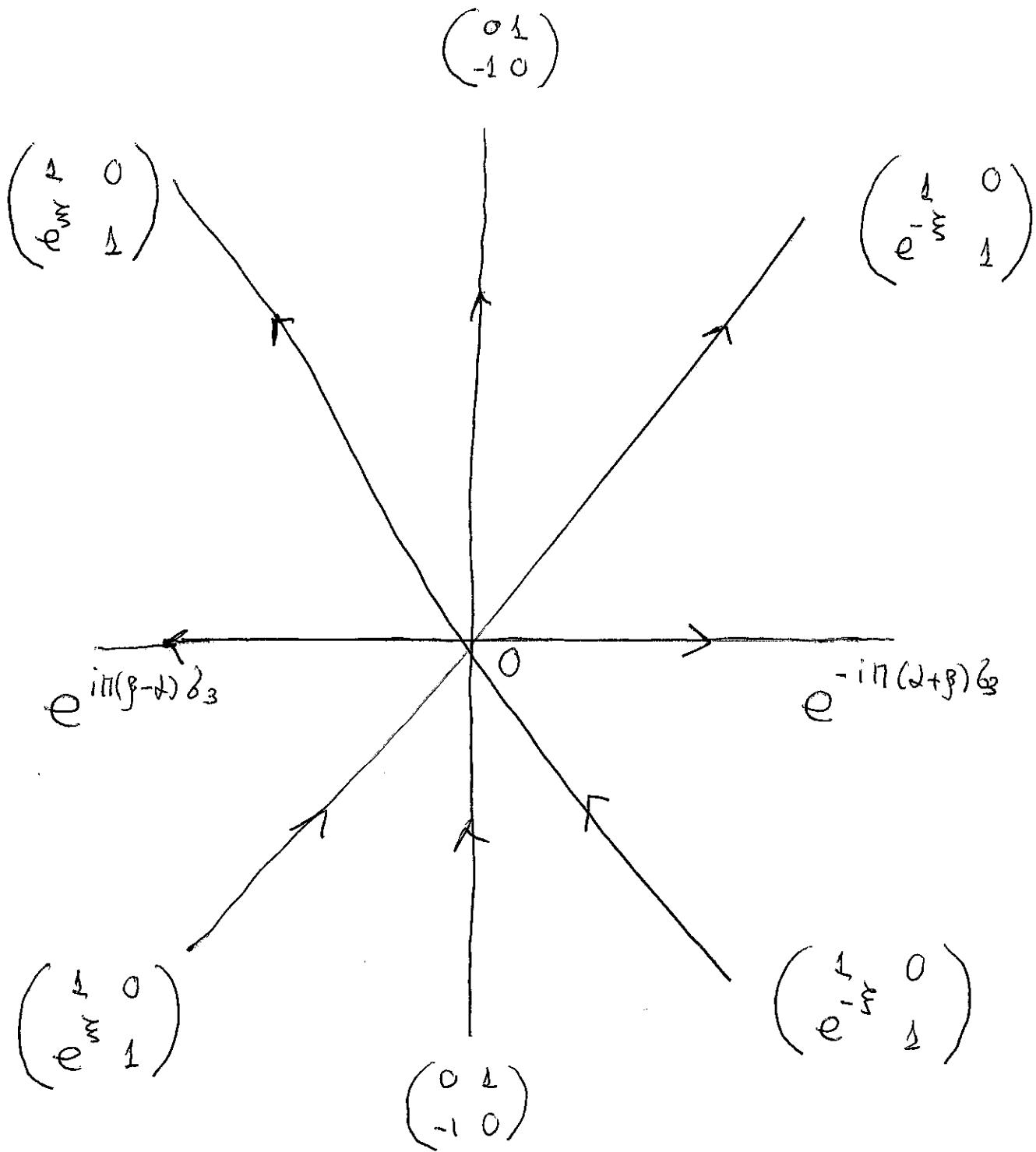


(2)

(5)

$\tilde{S}|_U \rightsquigarrow$

$\overset{(o)}{\mathcal{T}}(\xi) :$



$$(o) \quad (s) \quad (e)$$

$$\Phi(\xi) \rightarrow \Phi(\xi) = \Psi(\xi) \left\{ \begin{array}{l} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{||} \\ \text{I} \quad \text{||} \end{array} \right.$$

(1)  
 $\Psi(\xi)$ :

$$\begin{pmatrix} 1 & 0 \\ e^{i\pi} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -e^{\frac{i\pi}{2}\xi} \\ 0 & 1 \end{pmatrix}$$

$$e^{i\pi(\beta-\alpha)\xi}$$

$$e^{i\pi(\delta+\beta)\xi_3}$$

$$e^{i\pi(\alpha-\gamma)\xi}$$

$$e^{i\pi(\gamma-\delta)\xi_3}$$

$$\begin{pmatrix} 1 & 0 \\ e^{i\pi} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -e^{\frac{i\pi}{2}\xi} \\ 0 & 1 \end{pmatrix}$$

$$(1) \quad \Phi(\xi) \mapsto \begin{cases} (2) \quad \underline{\Phi}(\xi) = \begin{cases} (1) \quad \underline{\Phi}(\xi) \\ (2) \quad \underline{\Phi}(\xi) \end{cases} \end{cases} \left\{ \begin{array}{l} e^{i\pi(\alpha+\beta)\delta_3} \\ e^{i\pi(\alpha-\beta)\delta_3} \end{array} \right. \quad \begin{array}{l} \text{hatched rectangle} \\ \text{hatched rectangle} \end{array}$$

$$\underline{\Phi}(\xi) :$$

$$\begin{pmatrix} 1 & 0 \\ e^{\imath \xi} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -e^{-\imath \xi} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ e^{-2i\pi(\beta-\alpha)}e^{\imath \xi} & 1 \end{pmatrix}$$

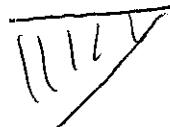
$$e^{-2i\pi\beta\delta_3}$$

$$\begin{pmatrix} 1 & -e^{-2i\pi(\alpha+\beta)}e^{-\imath \xi} \\ 0 & 1 \end{pmatrix}$$

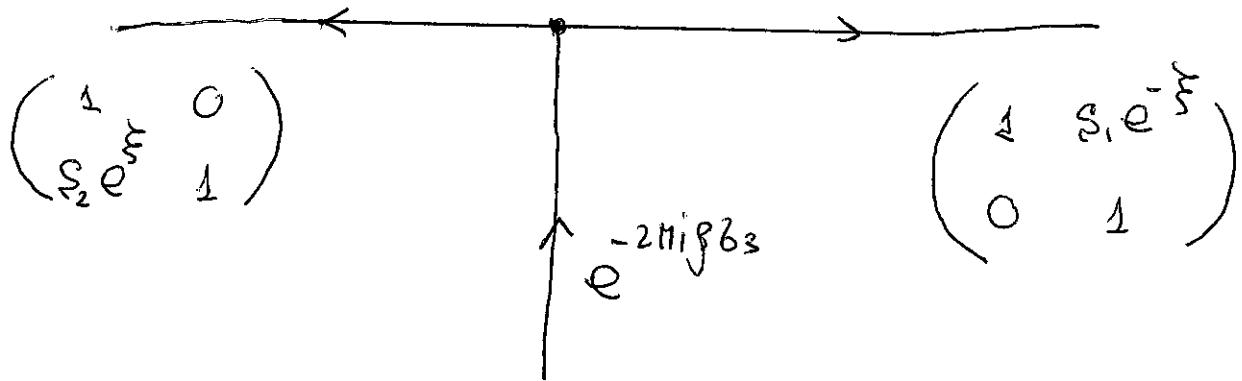
$$(2) \quad \Phi(\xi) \mapsto \tilde{\Phi}(\xi) = \overset{(2)}{\Phi}(\xi)$$

$$\left\{ \begin{array}{l} \left( \begin{array}{cc} 1 - e^{-\xi} & \\ 0 & 1 \end{array} \right) \quad \text{triangle} \\ \\ \left( \begin{array}{ccc} 1 & -e^{-2i\pi(\lambda+\beta)} & e^{-\xi} \\ & 0 & 1 \end{array} \right) \quad \text{triangle with wavy lines} \\ \\ \left( \begin{array}{cc} 1 & 0 \\ -e^{-\xi} & 1 \end{array} \right) \quad \text{triangle with diagonal lines} \end{array} \right.$$

$$\left( \begin{array}{cc} 1 & 0 \\ -e^{-2i\pi(\beta-\lambda)} & e^{-\xi} \\ & 1 \end{array} \right)$$



<sup>(3)</sup>  
 $\hat{f}(\xi) :$



$$S_1 = -2i e^{-i\pi(\delta+\beta)} \sin\pi(\delta+\beta), \quad S_2 = -2i e^{i\pi(\delta-\beta)} \sin\pi(\delta-\beta)$$

$$\hat{f}(\xi) = (I + O(\frac{1}{\xi})) \xi^{-\beta \zeta_3}, \quad \xi \rightarrow \infty$$

$$-\frac{\pi}{2} < \arg \xi < \frac{3\pi}{2}$$

$$\hat{f}(\xi) e^{-\frac{\xi}{\pi} \zeta_3} = \hat{f}(\xi) \xi^{\beta \zeta_3} C_j \quad j = 1, 2, 3$$

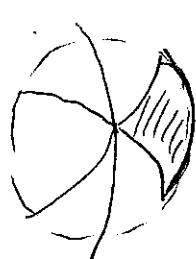
$\hat{f}(\xi)$  - holomorphic at  $\xi = 0$ .

## Comments:

- On the behavior at  $\xi = \infty$ :

$$\xi \sim \infty \Leftrightarrow z \sim \partial U$$

$$\tilde{S}|_{\partial U} \sim P^{(\infty)}(z) \varphi^{b_3/2}$$



$$\equiv (z-1)^{(g-2)b_3} (z-1)^{b_3}$$

up to holomorphic factors

$$= (z-1)^{gb_3} = n^{gb_3} \xi^{gb_3}$$

Hence:

$$\boxed{\text{(c)} \quad \tilde{\Phi}(\xi) \sim \xi^{gb_3}, \quad \xi \rightarrow \infty}$$

$$\stackrel{(1)}{\mathcal{F}}(\xi) = \stackrel{(0)}{\mathcal{F}}(\xi) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ in } \boxed{\quad}$$

(60)

Hence  $\stackrel{(1)}{\mathcal{F}}(\xi) \sim \xi^{-\beta_3}$  — and the same  
for all following  $\mathcal{F}$ -s. That is.

$$\boxed{\stackrel{(3)}{\mathcal{F}}(\xi) \sim \xi^{-\beta_3}, \xi \rightarrow \infty}$$

On the behavior at  $\xi = 0$ .

Roughly:  $\stackrel{(3)}{\mathcal{F}}(\xi) = O(\xi^{\pm 2})$

Let us show that the specification

$$\stackrel{(3)}{\mathcal{F}}(\xi) e^{-\frac{\xi}{2}\beta_3} = \stackrel{\wedge}{\mathcal{F}}(\xi) \xi^{2\beta_3} C;$$

is consistent with the jumps indicated on p. 58

We have

On :  $\xrightarrow{\quad}$

$$\overset{(3)}{I}_+(x) = \overset{1}{\int}(x) \int e^{\frac{x}{2}b_3} C_1 e^{\frac{x}{2}b_3}$$

$$= \overset{1}{\int}(x) \int e^{\frac{x}{2}b_3} C_3 e^{\frac{x}{2}b_3} \begin{pmatrix} 1 & s_i e^{-x} \\ 0 & 1 \end{pmatrix}$$

$\Downarrow$

$$C_1 = C_3 e^{\frac{x}{2}b_3} \begin{pmatrix} 1 & s_i e^{-x} \\ 0 & 1 \end{pmatrix} e^{-\frac{x}{2}b_3}$$

$$= C_3 \begin{pmatrix} 1 & s_i \\ 0 & 1 \end{pmatrix} = C_3 S_1$$

$$S_1 = \begin{pmatrix} 1 & s_i \\ 0 & 1 \end{pmatrix},$$

$C_3 = C_1 S_1^{-1}$

62.

On:



$$\stackrel{(3)}{f}_+(\xi) = \hat{f}(\xi) \xi^{\frac{d}{2} b_3} C_2 e^{\frac{\xi}{2} i \pi b_3}$$

$$= \stackrel{(3)}{f}_-(\xi) \begin{pmatrix} 1 & 0 \\ s_2 e^{i \pi} & 1 \end{pmatrix}$$

$$= \hat{f}(\xi) \xi^{\frac{d}{2} b_3} C_1 e^{\frac{\xi}{2} i \pi b_3} \begin{pmatrix} 1 & 0 \\ s_2 e^{i \pi} & 1 \end{pmatrix}$$



$$C_2 = C_1 S_2, \quad S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix}$$

On :



(63)

$$\stackrel{(3)}{f}_+(\xi) = \hat{f}(\xi) \xi_+^{2\beta_3} C_2 e^{\frac{\pi i}{2}\beta_3}$$

$$= \stackrel{(3)}{f}_-(\xi) e^{-2\pi i \beta_3} = \hat{f}(\xi) \xi_-^{2\beta_3} C_3 e^{\frac{\pi i}{2}\beta_3} e^{-2\pi i \beta_3}$$

$$\Downarrow \quad \xi_+^{2\beta_3} = e^{2\pi i 2\beta_3} \xi_-^{2\beta_3}$$

$$e^{2\pi i 2\beta_3} C_2 = C_3 e^{-2\pi i \beta_3}$$

or

$$e^{+2\pi i 2\beta_3} C_1 S_2 = C_1 S_1^{-1} e^{-2\pi i \beta_3} \quad (63.1)$$

equation on  $C_1$  !

Put

$$A := S_1^{-1} e^{-2\pi i \beta_3} S_2^{-1}$$

Then (63.1) becomes:

$$C_1 A = e^{2\pi i \beta_3} C_1$$

↓ key moment!

$$A = \begin{pmatrix} e^{-2\pi i \beta} + S_1 S_2 e^{2\pi i \beta} & -S_1 e^{2\pi i \beta} \\ -S_2 e^{2\pi i \beta} & e^{2\pi i \beta} \end{pmatrix}$$

must have  $e^{\pm 2\pi i \alpha}$  as its eigenvalues.

and, it does!

and.

67

$$C_1 = \begin{pmatrix} e^{\pi i(\beta+\alpha)} \frac{\sin \pi(\beta-\alpha)}{\sin 2\pi\alpha} & -e^{\pi i(\beta-\alpha)} \frac{\sin \pi(\beta+\alpha)}{\sin 2\pi\alpha} \\ 1 & -1 \end{pmatrix}$$

$\backslash$ -diagonal. {accepted arbitrariness}

Now, assume that we ~~have~~  
have solution  $\overset{(3)}{\mathcal{I}}$  - problem

and see how we can define the  
local parametrix  $P^{(1)}(z)$ .

Local parametrix for  $S(z)$

66

in  $U$  can be defined by

the formula

$$P^{(1)}(z) = E(z) \overset{(1)}{\mathcal{F}}(\xi(z)) \varphi^{-\frac{3}{2}}(z)$$

$$= E(z) \overset{(3)}{\mathcal{F}}(\xi(z)) \left\{ \dots \right\}^{-1} \varphi^{-\frac{3}{2}}(z)$$

$E(z)$  - holomorphic at  $z=1$

Put:  $E(z) = P^{(0)}(z) \lesssim g_3^{\frac{3}{2}}(z) \varphi(z)$  (66.1)

in 

We have:

(67)

$$P^{(n)}(z)$$

$$E(z) = e^{V+\beta_3} (z-1)^{(\alpha+\beta)\beta_3} e^{-i\pi(\alpha+\beta)\beta_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\varphi^{\beta_3/2}$$

$$\times h^{\beta\beta_3} (\ln z)^{\beta\beta_3} \times e^{\frac{V}{2}\beta_3} (z-1)^{\alpha\beta_3} z^{\frac{\beta-2}{2}\beta_3} e^{-i\pi\frac{\alpha+\beta}{2}\beta_3}$$

$\sum \beta\beta_3$

$$= e^{\frac{V+V-\beta_3}{2}} e^{-i\pi\frac{\alpha+\beta}{2}\beta_3} z^{\frac{\alpha-\beta}{2}\beta_3} h^{-\beta\beta_3}$$

$$\times (z-1)^{(\alpha+\beta)\beta_3} (\ln z)^{-\beta\beta_3} (z-1)^{-\alpha\beta_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

67'

Observe, that

$$(z-1)^{(2+\beta)b_3} (h_z)^{-\beta b_3} (z-1)^{-2b_3}$$

$$= \left( I + \sum_{k=1}^{\infty} c_k (z-1)^k \right) (z-1)^{-\beta b_3} (z-1)^{(2+\beta)b_3}$$

$$\times (z-1)^{-2b_3}$$

$$= I + \sum_{k=1}^{\infty} c_k (z-1)^k$$

and hence,

$$E(z) = \sum_{k=0}^{\infty} d_k (z-1)^k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence,  $E(z)$  defined in (66.1) is indeed holomorphic.

Second point.

$$n \rightarrow \infty, z \curvearrowright$$

$$P^{(1)}(z) = P^{(\infty)}(z) \xi^{\beta_3} \varphi^{b_{3/2}}(z) \overset{(3)}{\Phi}(\xi(z)) \tilde{\varphi}(z)$$

$$= P^{(\infty)}(z) \xi^{\beta_3} \varphi^{b_{3/2}} \left( I + O\left(\frac{1}{\xi(z)}\right) \right) \xi^{-\beta_3} \varphi^{-b_{3/2}}(z)$$

$$= P^{(\infty)}(z) \left( I + O\left(\frac{1}{n^{1-2|\text{Re } \beta|}}\right) \right)$$

and, this is uniform on the whole

$\partial U$

69

Under the assumption,

$$|\operatorname{Re} \beta| < \frac{1}{2},$$

the first transformation, i.e.

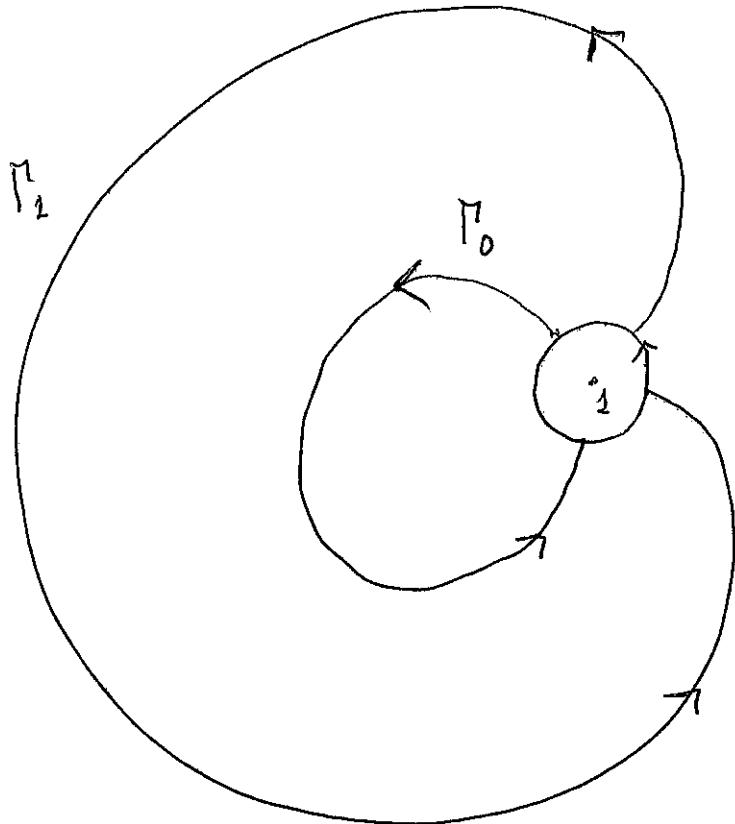
$$S \mapsto R$$

is:

$$R(z) = S(z) \times \begin{cases} [P^{(0)}(z)]^{-1} & \text{in } \text{Region U} \\ [P^{(0)}(z)]^{-1} & \text{in } \text{Region O} \end{cases}$$

and

R-problem:



$$G_R \Big|_{\partial U} = R_-^{-1} R_+ = \left[ P_{(z)}^{(\infty)} \right] \left[ P_{(z)}^{(1)} \right]^{-1}$$

$$= I + O\left(\frac{1}{h^{1-2l} \operatorname{Re} \beta l}\right)$$

$$G_R \Big|_{(\Gamma_0 \cup \Gamma_1) \setminus U} = I + O(e^{-\varepsilon n})$$



$$R(z) = I + O\left(\frac{z}{h^{1-2\Re p_1}(1+|z|)}\right)$$

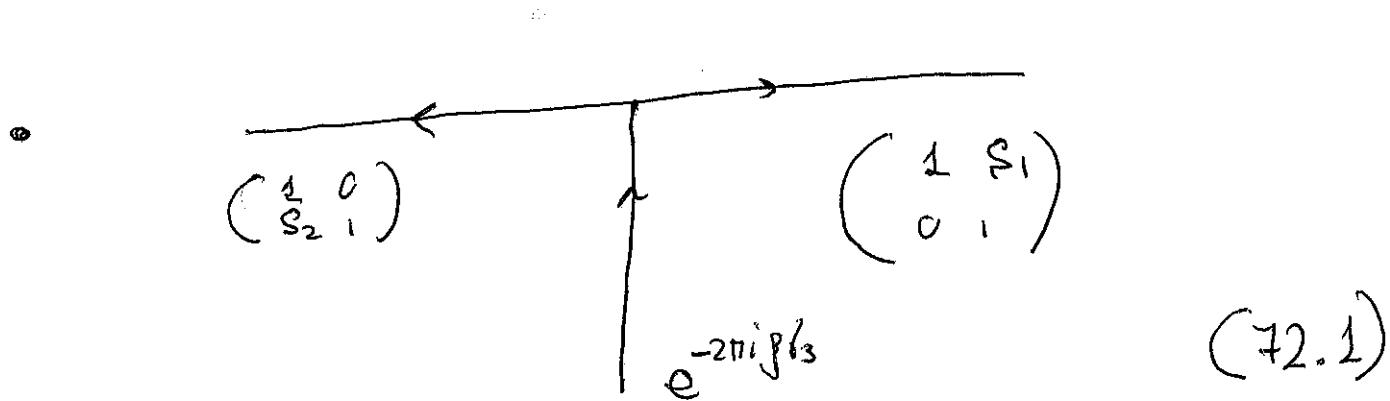
Now, how to solve  $\overset{(3)}{\mathcal{I}}$ -problem?

# Explicit solution of $\overset{(3)}{\mathfrak{L}}$ -problem.

(72)

$$\overset{(3)}{\Phi}(\xi) \mapsto \overset{(3)}{\Psi}(\xi) = \overset{(3)}{\Phi}(\xi) e^{-\frac{\xi}{2}\beta_3}$$

$\overset{(3)}{\Psi}(\xi)$ :



$$(1/s_2, 0) \quad (1/s_1, 0)$$

$$e^{-2\pi i \beta_3} \quad (72.1)$$

$$\overset{(3)}{\Psi}(\xi) = (\mathcal{I} + O(1/\xi)) \xi^{-\beta_3} e^{-\xi/2\beta_3}$$

$$\overset{(3)}{\Psi}(\xi) = \overset{\wedge}{\overset{(3)}{\Psi}}(\xi) \xi^{d\beta_3} C_j$$

! Remark

In our case,  $s_1, s_2$  are

specified as

$$s_1 = -2i e^{-i\pi(d+\beta)} \sin \pi(d+\beta)$$

$$s_2 = -2i e^{i\pi(d-\beta)} \sin \pi(d-\beta)$$

But, right now I am going to introduce:

$$\Psi(\xi) = \Psi(\xi; s_1, s_2)$$

and only assume that  $d, \beta$  - fixed and

then the only restriction on  $s_1, s_2$  is

that

$$[s_1 s_2 e^{2\pi i \beta} + 2 \cos 2\pi \beta = 2 \cos 2\pi d] \quad (73.1)$$

trace A =  $2 \cos 2\pi d$  - well posedness of  
 $\Psi$ -problem.  
 from p.64

! Claim.

$$\frac{d\Psi}{d\xi} = \left[ -\frac{1}{2}\mathcal{B}_3 + \frac{1}{N} A_0 \right] \Psi \quad (74.1)$$

for some  $A_0 = 2 \times 2$  matrix indep.  
on  $\xi$

Indeed, consider:

$$A(\xi) := \frac{d\Psi}{d\xi} \Psi^{-1}$$

We have:

$$\boxed{A_+(\xi) = A_-(\xi)} \Rightarrow$$

$$A(\xi) \in H(\mathbb{C} \setminus \{0\}).$$

{ note,  $\det \Psi \equiv 1$  ! }

$$\text{at } \xi = \infty : \quad \left( \Psi(\xi) = \left( I + \frac{m_1}{\xi} + \dots \right) \xi^{-\beta \zeta_3} e^{-\frac{\xi}{2} \zeta_3} \right)$$

(75)

$$A(\xi) = \left[ \left( I + \frac{m_1}{\xi} + \dots \right) \left( -\frac{1}{2} \zeta_3 - \frac{\beta \zeta_3}{m_1} \zeta_3 \right) + \dots \right]$$

$$\times \left( I - \frac{m_1}{\xi} + \dots \right)$$

$$= -\frac{1}{2} \zeta_3 + O\left(\frac{1}{\xi}\right)$$

$$\underset{\xi \rightarrow \infty}{\underset{\parallel}{\lim}} \left( -\frac{1}{2} [m_1, \zeta_3] - \beta \zeta_3 \right) + O\left(\frac{1}{\xi^2}\right)$$

$$\text{at } \xi = 0 :$$

$$A(\xi) = \frac{d \overset{\wedge}{\Psi}(0) \zeta_3 \overset{\wedge}{\Psi}'(0)}{\xi} + O(1)$$

Therefore:

$$A(\xi) = -\frac{1}{2} \beta_3 + \frac{1}{\xi} A_0$$

where for  $A_0$  we have:

$$A_0 = 2 \hat{\Psi}(0) \beta_3 \hat{\Psi}^{-1}(0)$$

$$= -g \beta_3 - \frac{1}{2} [m_1, \beta_3]$$

We have then:

$$A_0 = \begin{pmatrix} -\beta & \beta \\ c & \beta \end{pmatrix} \quad \beta = (m_1)_{12}$$

$$c = -(m_1)_{21}$$

$$\beta^2 + \beta c = d^2 \quad (76.1)$$

We have the a map

$$RH \rightarrow A(\varepsilon) = -\frac{1}{2} B_3 + \frac{1}{\varepsilon} A_0$$

$$(S_1, S_2) \rightarrow (B, C)$$

Inverse map:  $(B, C) \rightarrow (S_1, S_2)$  constitutes  
the Direct Monodromy Problem  
associated with equation (74.1),

$$\frac{d\Psi}{d\varepsilon} = \left[ -\frac{1}{2} B_3 + \frac{1}{\varepsilon} A_0 \right] \Psi$$

Consider the application of the  
general monodromy theory of  
linear systems to our case.

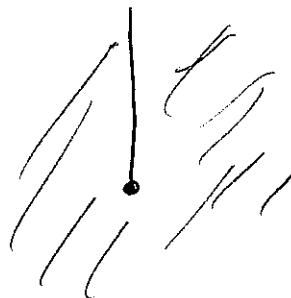
# Canonical Solutions

JP

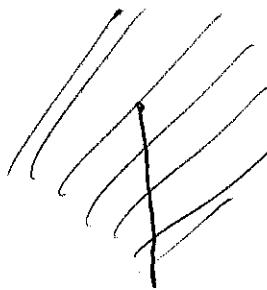
$$\Psi_k(\xi) = \left( I + \frac{m}{\xi} + \dots \right) e^{-\frac{\pi i}{N} \beta_3} \xi^{-g \beta_3}$$

$\xi \rightarrow \infty \quad , \quad k = 1, 2, 3 :$

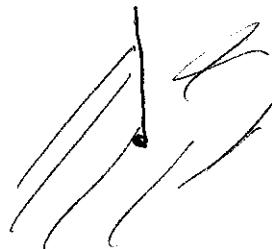
$k=1:$        $-\frac{3\pi}{2} < \arg \xi < \frac{\pi}{2}$



$k=2:$        $-\frac{\pi}{2} < \arg \xi < \frac{3\pi}{2}$



$k=3:$        $\frac{\pi}{2} < \arg \xi < \frac{5\pi}{2}$



$$\Psi_3(\xi) = \Psi_1(\xi e^{-2\pi i}) e^{-2\pi i \beta_3}$$

Also:

$$\overset{\circ}{\Psi}(\xi) = \overset{\wedge}{\Psi}(\xi) \xi^{26_3}$$

Stokes Matrices:

$$\cdot \quad \overset{\circ}{\Psi}_2(\xi) = \overset{\wedge}{\Psi}_1(\xi) S_1$$

$$\begin{array}{ccc} \diagup & \rightarrow & S_1 = \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\overset{\circ}{\Psi}_3(\xi) = \overset{\circ}{\Psi}_2(\xi) S_2$$

$$\begin{array}{ccc} \diagup & \rightarrow & S_2 = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \end{array}$$

Connection matrix:

$$\overset{\circ}{\Psi}_2(\xi) = \overset{\circ}{\Psi}(\xi) C$$

$\left\{ \text{out add } C_1 \right\}$

Cyclic relation:

$$C^{-1} e^{+2\pi i \beta_3} C = S_1^{-1} e^{-2\pi i \beta_3} S_2^{-1}$$

↓

(73.1):

$$S_1 S_2 e^{2\pi i \beta} + 2 \cos 2\pi \beta = 2 \cos 2\pi \delta$$

Now, having  $\Psi_{1c}(\xi)$ , we can define  
solution of  $\Psi$ -RH problem

$\Psi(\xi) :$

$\Psi_2(\xi) \quad 0 < \arg \xi < \pi$

$\Psi_3(\xi)$

$\pi < \arg \xi < \frac{3\pi}{2}$

$\Psi_1(\xi)$

$-\frac{\pi}{2} < \arg \xi < 0.$

$$\Psi_k(\xi) = \Psi_k(\xi; b, c)$$

$$\Psi(\xi) = \Psi(\xi; g_1, g_2)$$

need to know

$$(b, c) \longleftrightarrow (g_1, g_2)$$

Key fact:  $\Psi_k(\xi; b, c)$  can be found  
explicitly in terms of contour  
integrals.

Indeed:

$$\Psi_{11}^{(1)} = -\frac{1}{2}\Psi_{11} - \frac{3}{\xi}\Psi_{11} + \frac{6}{\pi}\Psi_{21}$$

$$\Psi_{21}^{(1)} = \frac{1}{2}\Psi_{21} + \frac{3}{\xi}\Psi_{21} + \frac{c}{\pi}\Psi_{11}$$

(82)

$$\Psi_{21} = \frac{\xi \Psi_{11}' + (\beta + \xi/2) \Psi_{11}}{6}$$

$$\xi \Psi_{11}'' + \Psi_{11}' + \left( \frac{1}{2} - \frac{\alpha}{4} - \frac{\alpha^2}{m^2} - \beta \right) \Psi_{11} = 0$$

$$\Psi_{11} := \xi^\alpha W(\xi)$$

$$\xi W'' + (2\alpha + 1) W' + \left( \frac{1}{2} - \beta - \frac{\xi}{4} \right) W = 0$$

in fact,  $W := e^{-\frac{\xi}{2}} V(\xi)$ .

$$\xi V'' + V' (2\alpha + 1 - \xi) - (\alpha + \beta) V = 0$$

CHP:  $\Psi(a, c; \xi)$ :

$$\xi \Psi' + (c - \xi) \Psi' - a \Psi = 0$$

$$\Psi_k \leftarrow \Psi(\alpha + \beta, 2\alpha + 1; \xi)$$

B



$S_1, S_2 = \text{explicit } (\Gamma\text{-function involved})$

formulas in terms of  $B, C$ .

or

$B, C = \text{explicit } (\Gamma\text{-function involved})$

formulas in terms of  $S_1, S_2$ .



Explicit  $P^{(1)}(z)$ .



Asymptotic of  $Y(z; n)$

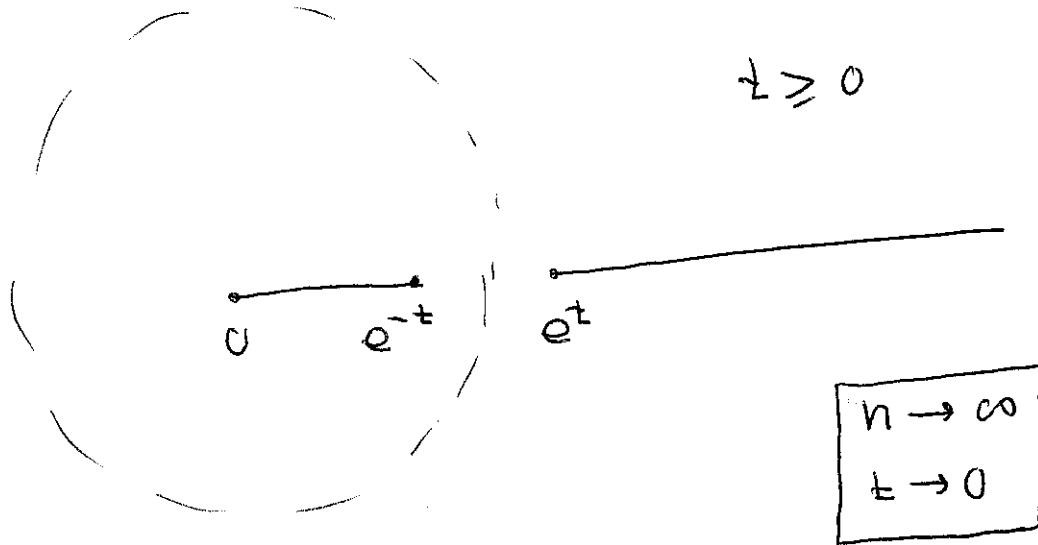
## 6. Transition asymptotics

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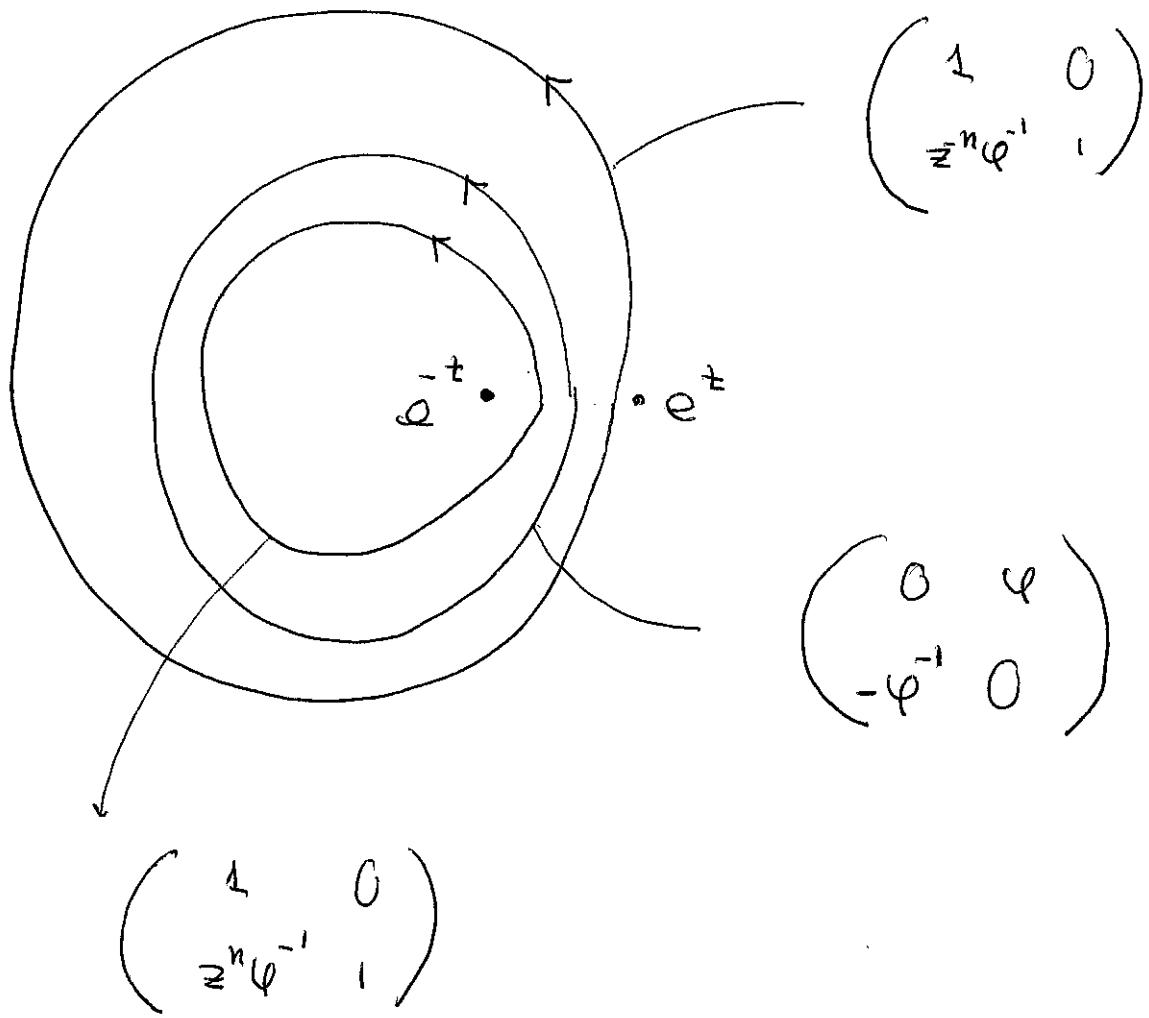
Painlevé V.

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$$\varphi(z) = (z - e^t)^{\alpha+\beta} (z - e^{-t})^{\alpha-\beta} z^{-\alpha+\beta} \\ \times e^{-i\pi(\alpha+\beta)} \frac{V(z)}{e}$$



$Y \rightarrow S$  as before:



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$$\varphi(z) = \varphi_-(z) \varphi_+(z)$$

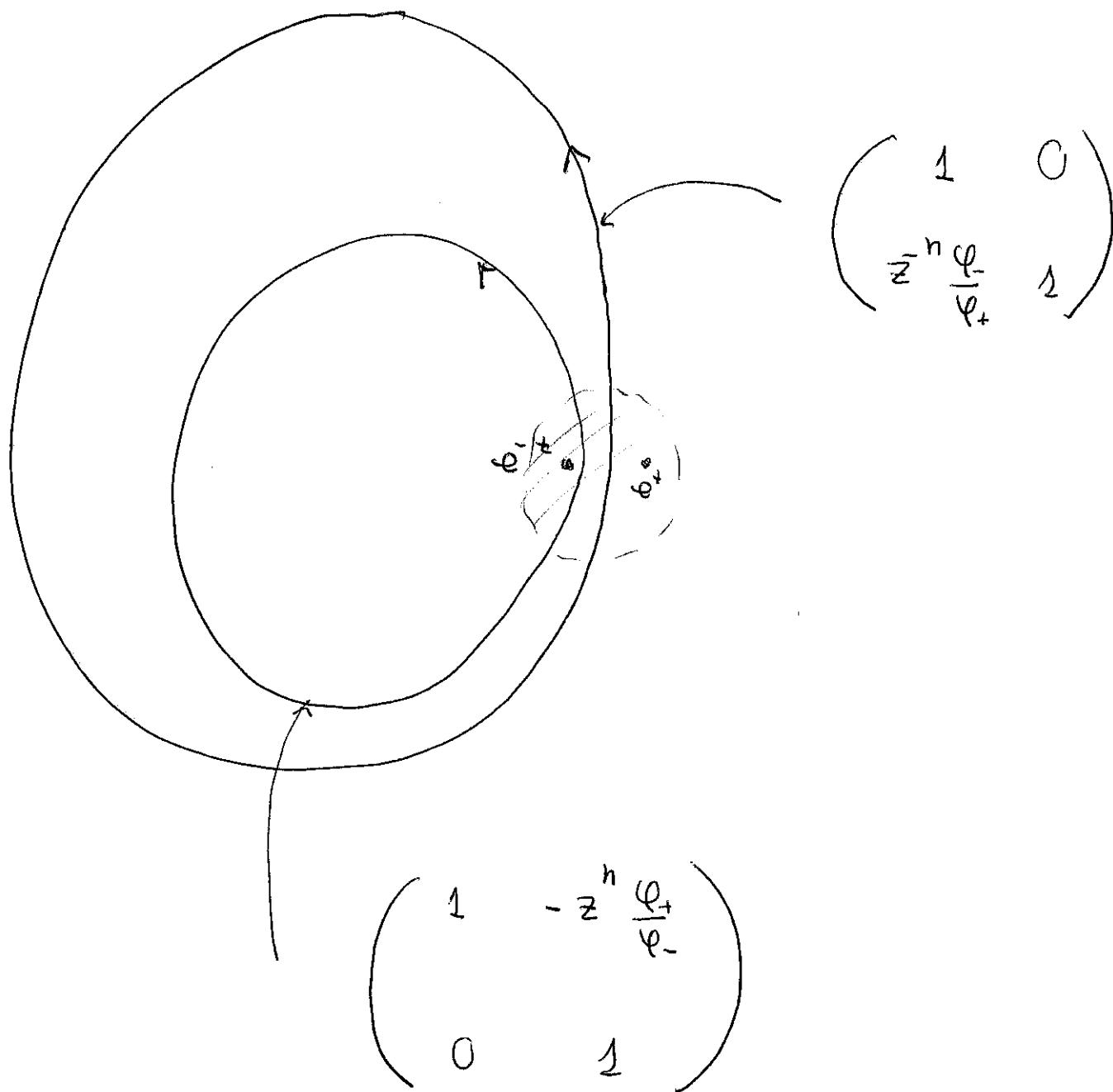
$$= \left[ \left( \frac{z - e^{-t}}{z} \right)^{\beta - \gamma} e^{V_-} \right] \left[ e^{V_+ (z - e^t)^{\beta + \gamma} - i\pi(\beta + \gamma)} \right]$$



$$P^{(0)}(z) = \begin{cases} \begin{pmatrix} \varphi_-^{-1} & 0 \\ 0 & \varphi_- \end{pmatrix} & |z| > 1 \\ \begin{pmatrix} 0 & \varphi_+ \\ -\varphi_+^{-1} & 0 \end{pmatrix} & |z| < 1 \end{cases}$$

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$$\mathcal{S} \rightarrow \mathcal{S}^2 = \mathcal{S} [P^{(0)}]^{-1}$$



$$\underline{V = 0}$$

$$\xi = n \ln z$$

$$\varphi_-(z) = \left( \frac{e^{\frac{\xi}{n}} - e^{-z}}{e^{\frac{\xi}{n}}} \right)^{d-\beta}$$

$$= \left( 1 - e^{-z - \frac{\xi}{n}} \right)^{d-\beta} = \left( 1 - e^{-\frac{\xi + \alpha/2}{n}} \right)^{d-\beta}$$

$$\boxed{x = 2 \pm n} \quad - \text{scaling parameter}$$

$$\approx n^{d-\alpha} \left( \xi + \frac{\alpha}{2} \right)^{d-\beta}$$

$$\psi_+(z) = (e^{\frac{\xi}{n}} - e^z)^{\alpha+\beta} e^{-i\pi(\alpha+\beta)}$$

$$= e^{-i\pi(\alpha+\beta) + \frac{\pi i}{n}(\alpha+\beta)} \left( 1 - e^{\frac{x-\xi}{n}} \right)^{\alpha+\beta}$$

$$\approx e^{-i\pi(\alpha+\beta)} n^{-\alpha-\beta} \left( \xi - \frac{x}{n} \right)^{\alpha+\beta}$$



For the local  $\mathbb{S}$ -model we have:

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$$-\frac{x_2}{2}$$

$$\frac{x_2}{2}$$

1

0

1

$$\frac{e^{-\xi} e^{i\pi(\alpha+\beta)} n^{2\beta}}{(\xi - \frac{x_2}{2})^{\alpha+\beta} (\xi + \frac{x_2}{2})^{\beta-\alpha}}$$

1

$$-e^{\xi} e^{-i\pi(\alpha+\beta)} \frac{(\xi - \frac{x_2}{2})^{\alpha+\beta} (\xi + \frac{x_2}{2})^{\beta-\alpha}}{n^{2\beta}}$$

0

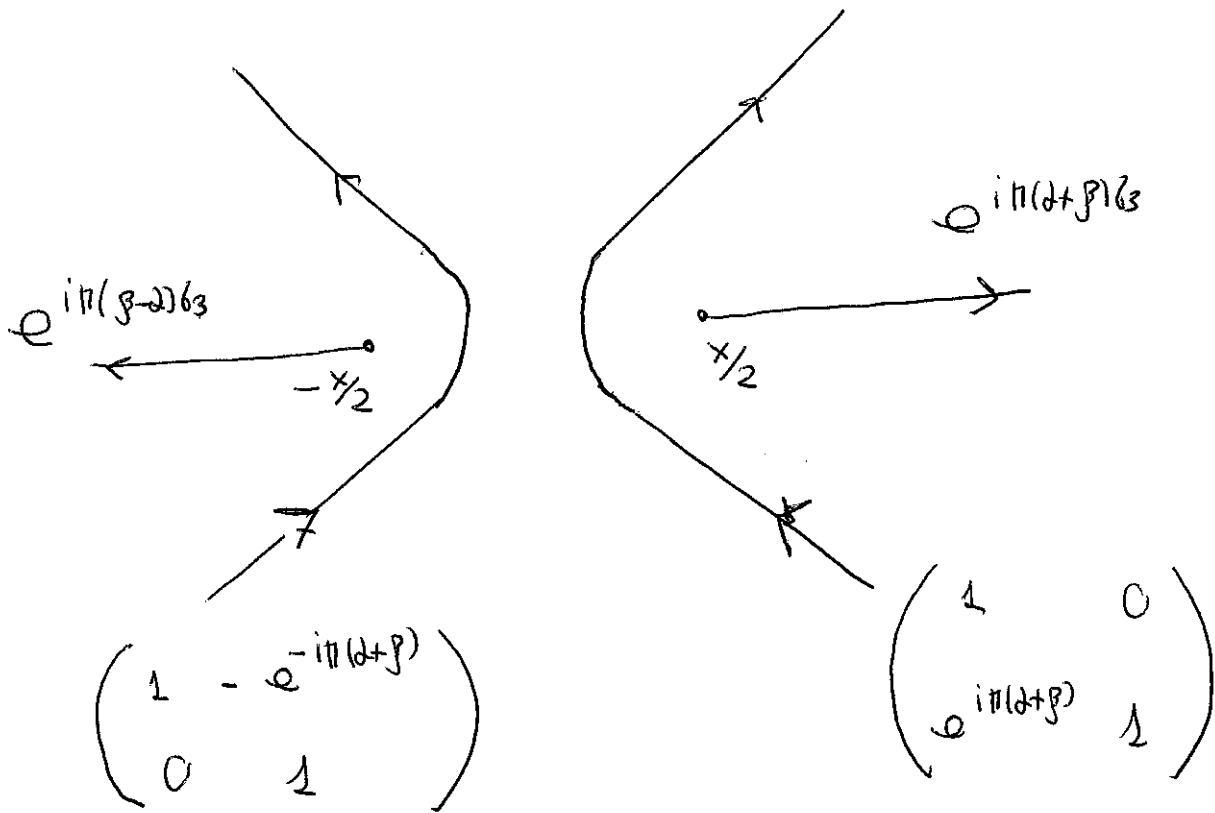
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g1

$$\overset{\circ}{\Psi}(\xi) := \overset{\circ}{\Phi}(\xi)$$

$$\times \exp \left\{ \left[ \frac{x}{2} + \frac{\beta-2}{2} \ln \left( \xi + \frac{x}{2} \right) + \frac{\beta+2}{2} \ln \left( \xi - \frac{x}{2} \right) - \beta \ln \eta \right] \zeta_3 \right\}$$

then,  $\overset{\circ}{\Psi}$ -problem:



$$\overset{\circ}{\Psi}(\xi) = (I + O(\gamma_\xi)) \sum e^{\frac{\beta b_3}{N} \xi} \zeta^{\pm}$$

$$\overset{\circ}{\Psi}(\xi) = \hat{\Psi}^{(\pm)}(\xi) \left( \xi \mp \frac{x}{2} \right)^{\frac{\beta+2}{2} b_3} \quad \xi \approx \pm \frac{x}{2}.$$

↓

$$\frac{d\psi^0}{d\xi} = \left[ \frac{1}{2} \beta_3 + \frac{A_+}{\xi - \xi_2} + \frac{A_-}{\xi + \xi_2} \right] \psi^0$$



$$A_+, A_- \equiv \{ y_{p\bar{v}} \}$$

$$\frac{d\psi^0}{dx} = \frac{1}{2} \left[ \frac{A_-}{\xi + \xi_2} - \frac{A_+}{\xi - \xi_2} \right] \psi^0$$