§ Categories of correspondences

char $k = 0$, oriented cohomology theories $:=$ free theories $\Omega^* \otimes_{\mathbb{L}} R$

where Ω^* – algebraic cobordism of Levine–Morel

$\mathbb{L}$ – Lazard ring, $\mathbb{L} \xrightarrow{\sim} \Omega^*(\mathrm{Spec}\, k)$

and $\mathbb{L} \longrightarrow R$ – morphism of graded (comm.) rings.

$$R^* := \Omega^* \otimes_{\mathbb{L}} R \;:\; Sm_k^{op} \longrightarrow CRing^*$$

$$f: X \to Y \rightsquigarrow f^*: R^*(Y) \to R^*(X)$$

$$g: Z \longrightarrow T \quad g_*: R^*(Z) \to R^{*-c}(T)$$

projective of rel. dim. c

Examples: $CH^* = \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$ $\qquad \mathbb{L} \to \mathbb{L}/\mathbb{L}^{<0} \xrightarrow{\sim} \mathbb{Z}$

$K_0[\beta, \beta^{-1}] = \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ $\qquad \mathbb{L} \longrightarrow \mathbb{Z}[\beta, \beta^{-1}]$, $[X] \mapsto \beta^{\dim X} \cdot \chi(\mathcal{O}_X)$

Algebraic cobordism possess maximum information on oriented coh. theories, but what about motives?

R^* - oriented coh. theory, we define

the category $\mathrm{Corr}_R(k)$: $\mathrm{Ob} = \mathrm{Ob}\ \mathrm{SmProj}_k$

Mor : $\mathrm{Hom}_{\mathrm{Corr}}(X,Y) := R^{\dim Y}(X \times Y)$

composition of correspondences $\quad X \overset{\alpha}{\dashrightarrow} Y \overset{\beta}{\dashrightarrow} Z \qquad \alpha \in R^{\dim Y}(X \times Y)$, $\beta \in R^{\dim Z}$

$$\beta \circ \alpha := \pi_{XZ*}\left(\pi_{XY}^*(\alpha) \cdot \pi_{YZ}^*(\beta)\right)$$

$$\begin{aligned} \mathrm{SmProj}_k &\longrightarrow \mathrm{Corr}_R(k) \\ X &\longmapsto X \\ X \xrightarrow{f} Y &\longmapsto [\Gamma_f] \in R^{\dim Y}(X \times Y) \end{aligned}$$

$\Gamma_f \overset{i}{\hookrightarrow} X \times Y$

$i_* 1_{\Gamma_f} =: [\Gamma_f]$

$\mathrm{Hom}_R(X,Y)$ are "universal transforms" from cohomology of Y to cohomology of X

E.g. $\exists\, \mathrm{Re}_R$: $\mathrm{Corr}_R(k)^{op} \longrightarrow \mathrm{Mod}_R^*$

$$\begin{aligned} X &\longmapsto R^*(X) \\ X \overset{\alpha}{\dashrightarrow} Y &\longmapsto R^*(Y) \xrightarrow{\alpha^*} R^*(X) \end{aligned}$$

$$\alpha^*(y) := \pi_{X*}(\alpha \cdot \pi_{Y*}(y))$$

Realization functors often take values in (idempotent-complete) abelian categories

We can formally add idempotents to $\mathrm{Corr}_R(k)$:

new objects : (X, π) ; $X \in \mathrm{SmProj}_k$, $\pi \in R^{\dim X}(X \times X)$

$\pi \circ \pi = \pi$.

$$\mathrm{Hom}((X,\pi),(Y,p)) = p \circ \mathrm{Hom}_{\mathrm{Corr}}(X,Y) \circ \pi .$$

Moreover, $\mathrm{Corr}_R(k)$ has monoidal structure

$$X \otimes Y := X \times Y$$

and realization functors are monoidal taking the Tate object $\mathbb{Z}_R(1)$ to a dualizable object

$\mathrm{Spec}\, k \xrightarrow{x} \mathbb{P}^1_k \longrightarrow \mathrm{Spec}\, k$ gives an idempotent $\underbrace{}_{\pi_{\mathbb{Z}(0)}}$ of $\mathbb{P}^1_k$ in Corr

any point

define $\mathbb{Z}_R(1) := (\mathbb{P}^1_k, \Delta - \pi_{\mathbb{Z}(0)})$.

The category of R^*-motives $\mathrm{PM}_R(k)$:

is idempotent completion + inverting $\mathbb{Z}_R(1)$ in $\mathrm{Corr}_R(k)$

Ob : (X, π, n), $n \in \mathbb{Z}$

$$\mathrm{Hom}((X,\pi,n),(Y,p,m)) := p \circ R^{\dim Y + m - n}(X \times Y) \circ \pi$$

We denote $M_R : \mathrm{SmProj}_k \longrightarrow \mathrm{PM}_R(k)$ | $M_R(X) = (X, \Delta_X, 0)$

Examples:

- $$M_R(\mathbb{P}^n) \simeq \overset{\pi_0}{\mathbb{Z}_R(0)} \oplus \overset{\pi_1}{\mathbb{Z}_R(1)} \oplus \dots \oplus \overset{\pi_n}{\mathbb{Z}_R(n)}$$

  where $\mathbb{Z}_R(k) := \mathbb{Z}_R(1)^{\otimes k}$.

E.g. if $R = CH^*$, $CH^n(\mathbb{P}^n \times \mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Z} \cdot z_1^i z_2^{n-i}$

$$[\Delta_{\mathbb{P}^n}] = \sum_{i=0}^{n} \underbrace{z_1^i z_2^{n-i}}_{\pi_i}$$

- $$\underset{\substack{\text{split}\\ \text{quadric}\\ \text{of dim. } d}}{M_R(\overline{Q}^d)} \simeq \begin{cases} \mathbb{Z}_R(0) \oplus \mathbb{Z}_R(1) \oplus \dots \oplus \mathbb{Z}_R(d), & \text{if } d \text{ is odd} \\ \bigoplus_{i=0}^{d} \mathbb{Z}_R(i) \oplus \mathbb{Z}_R(d/2), & \text{if } d \text{ is even} \end{cases}$$

In particular, $M_R(\mathbb{P}^{2n+1}) \simeq M_R(\overline{Q}^{2n+1})$
even though $\mathbb{P}^{2n+1} \not\simeq \overline{Q}^{2n+1}$ for $n > 0$.

- $M_{K_0[\beta,\beta^{-1}]}(Q)$ is of Tate type if disc & Clifford are trivial

- $M_{CH}(Q)$ can be indecomposable

Note that if $A^* \to B^*$ - morphism of theories, then $PM_A \to PM_B$.

Th (Vishik - Yagita)

The canonical functor $PM_\Omega(k) \to PM_{CH}(k)$

'has nilpotent kernel', allows lifts of idempotents,

$$\mathrm{Irr}\, \mathrm{Ob}\, PM_\Omega(k) \xrightarrow{\sim} \mathrm{Irr}\, \mathrm{Ob}\, PM_{CH}(k).$$

Remark. Even though $M_\Omega(\mathbb{P}^n_k)$ is a sum of Tate motives,

the decomposition is not at all unique, because

$$\mathrm{Hom}(\mathbb{Z}_\Omega(i), \mathbb{Z}_\Omega(j)) = \mathbb{L}^{j-i} \neq 0 \text{ if } j < i.$$

Functors of Riemann-Roch type

$(A, F_A) \xrightarrow{(\varphi, \gamma)} (B, F_B)$ - morphism of FGLs

$\varphi : A \to B$, $\gamma \in B[[x]]$

$\varphi(F_A)(\gamma(x), \gamma(y)) = \gamma F_B(x, y)$

If $\gamma(x) = b_0 x + b_1 x^2 + \ldots$ where $b_0 \in B^\times$

then Panin-Smirnov construct a multiplicative operation

$$A^* \xrightarrow{\bar{\varphi}} B^*$$

Moreover, they prove a Riemann-Roch result:

$$X \xrightarrow{f} Y \qquad \varphi(f_* x) = f_*\big(\varphi(x)\cdot Td_\gamma(T_X - f^*T_Y)\big)$$

projective morphism

where $Td_\gamma(t) := \dfrac{t}{\gamma(t)}$

$$Td_\gamma(L) := Td_\gamma(c_1(L))$$

$Td_\gamma(\underset{\substack{\cap \\ K_0(X)}}{\mathcal{U}})$ is computed by the splitting principle.

Clearly, $\bar{\varphi} : A^* \to B^*$ does not induce $PM_A \to PM_B$ by setting $A^{\dim Y}(X\times Y) \xrightarrow{\bar{\varphi}} B^{\dim Y}(X\times Y)$

<u>But</u> it does induce $\Phi_{(\varphi,\gamma)} : PM_A \to PM_B$

$$\begin{array}{ccc} A^{\dim Y}(X\times Y) & \longrightarrow & B^{\dim Y}(X\times Y) \\ \alpha & \longmapsto & Td^u(T_X)\cdot \bar{\varphi}(\alpha)\cdot Td^{1-u}(T_Y) \end{array}$$

where u is a number so that $(1+x)^u \in B[[x]]$,

e.g. $u=0$ or 1 works.

We choose $u = 0$, because in this case

$$\begin{tikzcd} PM_A(k) \arrow[rr, "\Phi_{(\varphi,\gamma)}"] & & PM_B(k) \\ & SmProj_k \arrow[ul, "M_A"] \arrow[ur, "M_B"'] & \end{tikzcd}$$

commutes.

These are <u>functors of Riemann-Roch type</u>, and we get a (strict) functor $FGL^{iso} \longrightarrow Cat$

$$(A, F_A) \longmapsto PM_A.$$

Eventually, our goal is to compute the limit of this functor.

There is the universal "invertible" multiplicative operation

$$\Omega^* \xrightarrow{S_{L-N}^{tot}} \Omega^*[b_0, b_0^{-1}, b_1, b_2, \ldots]$$

that corresponds to $(\mathbb{L}, F_{univ}) \xrightarrow{(t,\gamma)} (\mathbb{L}B, F_{univ})$

$$\gamma(x) := b_0 x + b_1 x^2 + \ldots$$

$$t(F_{univ})(x,y) = \gamma F_{univ}(\gamma^{-1}(x), \gamma^{-1}(y)).$$

<u>Lemma.</u> $(\mathbb{L}, F_{univ}) \underset{(s,x)}{\overset{(t,\gamma)}{\rightrightarrows}} (\mathbb{L}B, F_{univ})$ is initial in FGL.

where $s : \mathbb{L} \to \mathbb{L}B$ is the canonical map

Corollary: $\underset{FGL}{\mathrm{Lim}}\ PM \simeq \mathrm{Lim}\left(PM_\Omega \overset{F_1}{\underset{F_2}{\rightrightarrows}} PM_{\Omega B}\right)$.

This category is called the category of (pre) Landweber-invariant motives $PM_{inv}^{pre}(k)$

Def. An object in this limit is (X, π, r) such that $F_1\pi = F_2\pi$.

We denote by $M(X) = (X, \Delta_X, 0)$,

then
$$\mathrm{Hom}(M(X), M(Y)) := \{ f : M_\Omega(X) \to M_\Omega(Y) \mid F_1 f = F_2 f \}$$
$$= \{ \alpha \in \Omega^{\dim Y}(X \times Y) \mid S^{tot}_{L-N}(\alpha) = \alpha \cdot Td^{-1}(T_Y) \}$$

Clearly, $M : SmProj_k \longrightarrow PM^{pre}_{inv}(k)$ is a functor,

moreover, functors of Riemann-Roch type are monoidal,

and $PM^{pre}_{inv}(k)$ inherits a monoidal structure.

However, $PM^{pre}_{inv}(k)$ has no duals of $M(X)$!

In $PM_R(k)$ we have

$$\begin{array}{ccc} \mathrm{Hom}(M_R(X), M_R(Y)) & \simeq & \mathrm{Hom}\big(M_R(Y)(-\dim Y), \\ & & \quad M_R(X)(-\dim X)\big) \\ \| & & \| \\ R^{\dim Y}(X \times Y) & & R^{\dim X - \dim X - (-\dim Y)}(Y \times X) \end{array}$$

so, $M_R(X)^\vee$ is $M_R(X)(-\dim X) = M_R(X) \otimes \mathbb{Z}_R(1)^{\otimes(-\dim X)}$

$$= (X, \Delta_X, -\dim X).$$

However, we have broken the symmetry in correspondences in PM^{pre}_{inv}: $\mathrm{Hom}(M(X), M(Y))$ is a subgroup in $\Omega^*(X \times Y)$ that depends on $Td(T_Y)$ ...

Let's add more objects to PM_R and eventually to PM^{pre}_{inv}:

Instead of SmProj$_k$ let's start with $(X, V) \in K_0(X)$, $X \in$ SmProj$_k$

$$\mathrm{Hom}_{Corr_R}((X,V),(Y,W)) := R^{\dim Y + \mathrm{rk}\, W - \mathrm{rk}\, V}(X \times Y)$$

Well, this doesn't change the category,
(up to equivalence)

since $(X,V) \simeq X \otimes \mathbb{Z}(\operatorname{rk} V)$

but we also re-define functors of R.-R. type:

$$\Phi_{(\varphi,f)} : \operatorname{Hom}_{PM_A}((X,V),(Y,W)) \to \operatorname{Hom}_{PM_B}((X,V),(Y,W))$$

$$\begin{array}{c} \| \\ A^{\dim Y + \operatorname{rk} W - \operatorname{rk} V}(X \times Y) \\ \Psi \\ \alpha \longmapsto \underbrace{\bar{\varphi}(\alpha) \cdot Td(T_Y)}_{\text{old functor}} \cdot Td^{-1}(W) \cdot Td(V) \end{array}$$

(as before I omit pullbacks from X and Y)

// Motivation for (X,V):

$$\begin{tikzcd} X \times_V X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "0"] \\ X \arrow[r, hook, "0"] & V \end{tikzcd}$$

$X \times_V X \in dQSmProj_k$ derived scheme

$$\Omega_*(X \times_V X) \xrightarrow[\sim]{\pi_{1*}} \Omega_*(X)$$

because $t(X \times_V X) = X$

but $X \times_V X \to X$ has the same normal bundle as X in V //

Anyways, it is now easy to see that $M(X)^\vee$ is $M(X, -T_X)$:

$$\mathrm{Hom}(M(X), M(Y)) = \{\alpha \in \Omega^{\dim Y}(X \times Y) \mid S^{tot}_{L-N}(\alpha) = \alpha \cdot Td^{-1}(T_Y)\}$$

$$\mathrm{Hom}(M(Y, -T_Y), M(X, -T_X))$$

Then PM_{inv} is the limit of these extended $PM_\Omega \rightrightarrows PM_{\Omega B}$,

and it has strong duals of all objects.

<u>Realization functor on PM_{inv}:</u>

$$Re : PM_{inv}^{op} \longrightarrow QCoh(\mathcal{M}_{fg}) = (\mathbb{L}, \mathbb{LB})\text{-comod}$$

$$X \longmapsto \Omega^*(X)$$ with the action of the total Landweber-Novikov operation

$$\alpha \in \Omega^{\dim Y}(X \times Y) \rightsquigarrow \Omega^*(Y) \to \Omega^*(X)$$

$$u \longmapsto \pi_{X*}(\alpha \cdot \pi_Y^*(u))$$

$$S_{L-N}(\pi_{X*}(\alpha \cdot \pi_Y^*(u))) = \pi_{X*}(S_{L-N}(\alpha) \cdot \pi_Y^*(S_{L-N}(u)) \cdot Td(T_Y))$$

$$= \pi_{X*}\left(\hat{S}_{L-N}(u)\cdot\alpha\right) = \alpha\circ\left(S_{L-N}(u)\right)$$

$$S_{L-N}(\alpha) = \alpha\cdot Td^{-1}(T_Y)$$

<u>Examples:</u> $M(\mathbb{P}^d) = M(\operatorname{Spec} k) \oplus \underbrace{\tilde{M}(\mathbb{P}^d)}_{\text{indecomposable!}}$

$\operatorname{Spec} k \to \mathbb{P}^d \to \operatorname{Spec} k$

But $\Omega^*(\mathbb{P}^d) = \mathbb{L}\cdot 1 \oplus \underbrace{\bigoplus_{i=1}^{d} \mathbb{L} z^i}_{\text{irreducible } (\mathbb{L},\mathbb{LB})\text{-comodule}}$

$S(z) = b_0 z + b_1 z^2 + \dots$

$M(\overline{Q}^d) = M(\operatorname{Spec} k) \oplus \underbrace{\tilde{M}(\overline{Q}^d)}_{\text{indecomposable}}$

even if d is odd $\tilde{M}(\overline{Q}^d) \not\cong \tilde{M}(\mathbb{P}^d)$.

„<u>Th</u>": consider a subcategory $TM(k) \subset PM_{inv}(k)$ of motives such that their Ω-realization (or CH-realization) is Tate

Then $TM(k) \xrightarrow{Re} (\mathbb{L},\mathbb{LB})$-comod is fully faithful.

In general, this is of course very far from being true.

More properties:

- Functor

$$\begin{tikzcd} PM_{inv} \arrow[r, hook] \arrow[dr, two heads, "Re_{CH}"] & PM_{\Omega} \arrow[d] \\ & PM_{CH} \end{tikzcd}$$

  - Re_{CH} is conservative
  - $Re_{CH} \otimes \mathbb{Q} : PM_{inv} \otimes \mathbb{Q} \xrightarrow{\sim} PM_{CH} \otimes \mathbb{Q}$ is an equivalence
  - Re_{CH} restricted to TM is faithful

Remark:

$$\begin{tikzcd} PM_{inv} \otimes \mathbb{Q} \arrow[r, "Re"] \arrow[d, "\wr"] & (\mathbb{L}\otimes\mathbb{Q}, \mathbb{LB}\otimes\mathbb{Q})\text{-comod} \arrow[d, "\wr"] \\ PM_{CH} \otimes \mathbb{Q} \arrow[r, "Re_{CH}"] & \mathbb{Q}\text{-Vect} \end{tikzcd}$$

- equivalence

We had a strict functor FGb $\to$ Cat and have computed its limit, but from categorical perspective this might not be the best thing to do (Cat is 2-category & limits do not respect equivalences of cat's)

2-limit is a notion of limit natural for 2-categories.

basically we allow diagrams to commute up to 2-morphism

$$\begin{tikzcd}
\text{2-Lim} \arrow[r, squiggly] \arrow[dr, bend right] & PM_{\Omega} \arrow[d] \arrow[dl, phantom, "\Downarrow \xi"] \\
 & PM_{CH}
\end{tikzcd}$$

ξ is an invertible natural transformation

but also these 2-morphisms have to satisfy some coherence

$$\begin{tikzcd}
t \arrow[r, bend left] \arrow[dr] \arrow[ddr, bend right] & i \arrow[d, "u"] \arrow[dl, phantom, "\Downarrow \xi_u"] \\
 & j \arrow[d, "v"] \arrow[dl, phantom, "\Downarrow \xi_v"] \\
 & k
\end{tikzcd}$$

$$\xi_{v \circ u} = \xi_v \circ \xi_u$$

computing limits in Sets is easy

and so is computing 2-limits in Cat:

$$\underset{I \xrightarrow{F} Cat}{2\text{-Lim}(F)} : \quad Ob = \Big(\underset{Ob\,F(i)}{x_i \in}, \quad x_i \xrightarrow{\sim} F(u)(x_j) \quad j \xrightarrow{u} i \Big)$$

satisfying compositions ...

$$Mor = \quad \cdots$$

We have $PM_{-} : FGL^{iso} \longrightarrow Cat$

its 2-limit is the category of Landweber-equivariant motives $\quad PM_{equiv}$

Lemma. There is a "2-initial" diagram in FGL

$$(\mathbb{L}, F_{univ}) \underset{(t, f)}{\overset{(s, x)}{\rightrightarrows}} (\mathbb{L}B, F_{univ}) \xrightarrow[(\pi_{13},\, x)]{(\pi_{12},\, x),\ (\pi_{23},\, f_{23})} (\mathbb{L}B \times B', F_{univ})$$

$$\pi_{12} : \mathbb{L}B \longrightarrow \mathbb{L}B \times B'$$

$$\lambda b \longmapsto \lambda b$$

$$\pi_{23} : \quad \lambda b \longmapsto S'(\lambda)\, b'$$

$$\pi_{13} : \quad \lambda b \longmapsto \lambda b'$$

Where does this come from?

$$\begin{tikzcd}
& \operatorname{Spec} L[B\times B'] \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec} LB \arrow[dl] \arrow[dd] \\
\operatorname{Spec} LB \arrow[rr] \arrow[dd] & & \operatorname{Spec} L \arrow[dd] & \\
& \operatorname{Spec} LB \arrow[rr] \arrow[dl] & & \operatorname{Spec} L \arrow[dl] \\
\operatorname{Spec} L \arrow[rr] & & \mathcal{M}_{fg} &
\end{tikzcd}$$

a morphism in FGL is a morphism here over $\underline{\mathcal{M}_{fg}}$.

i.e. together with 2-morphism

$$\begin{tikzcd}
\operatorname{Spec} LB \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\Swarrow \gamma"] & \operatorname{Spec} L \arrow[d] \\
\operatorname{Spec} L \arrow[r] & \mathcal{M}_{fg}
\end{tikzcd}$$

We choose an FGL on all of the schemes above, i.e. we choose the coordinate.

This leads to these strange morphisms in FGL.

<u>Cor.</u> If we denote the image of this diagram in Cat as follows

$$PM_{\Omega} \underset{F_2}{\overset{F_1}{\rightrightarrows}} PM_{\Omega B} \begin{array}{c} \xrightarrow{G_{12}} \\ \xrightarrow{G_{23}} \\ \xrightarrow[G_{13}]{} \end{array} PM_{\Omega B\times B'}$$

then PM_{equiv} : $Ob = \left(\underset{PM_{\Omega}}{\underset{\cap}{M}}, \gamma_M : F_1 M \xrightarrow{\sim} F_2 M \right)$ such that

$$G_{13}(\gamma_M) = G_{23}(\gamma_M) \circ G_{12}(\gamma_M)$$

$$\operatorname{Hom}((M,\gamma_M),(N,\gamma_N)) = \left\{ M \xrightarrow{\varphi} N \text{ in } PM_{\Omega} \;\middle|\; \begin{tikzcd} F_1M \arrow[r, "F_1\varphi"] \arrow[d, "\wr\gamma_M"'] & F_1N \arrow[d, "\wr\gamma_N"] \\ F_2M \arrow[r, "F_2\varphi"'] & F_2N \end{tikzcd} \right\}$$

we have used equalities of functors $G_{12} \circ F_1 = G_{13} \circ F_1$ and so on.

<u>Prop.</u> $PM_{inv} \hookrightarrow PM_{equiv}$ is a fully faithful functor

which contains motives of smooth projective varieties and their duals.

$\triangleright$ Basically, invariant motives are such that $F_1 M = F_2 M$: $\phi_M = id$

clearly it satisfies the cocycle condition

$$\mathrm{Hom}\left((M, id_{F_1 M = F_2 M}), (N, id_{F_1 N = F_2 N})\right) =$$

$$= \left\{ M \xrightarrow{\varphi} N \;\middle|\; \begin{tikzcd} F_1 M \arrow[r, "F_1\varphi"] \arrow[d, no head] & F_1 N \arrow[d, no head] \\ F_2 M \arrow[r, "F_2\varphi"'] & F_2 N \end{tikzcd} \text{ i.e. } F_1\varphi = F_2\varphi \right\}$$

$\square$

Also, it is easy to see that PM_{equiv} is monoidal, and every object has a dual.

Exact structure on PM_{equiv}

Trivial exact structure on PM_Ω:

$$0 \to (X, \pi_X, r) \xrightarrow{i} (Y, \pi_Y, \ell) \xrightarrow{p} (Z, \pi_Z, k) \to 0$$

is exact iff $(Y, \pi_Y, \ell) \simeq (X, \pi_X, r) \oplus (Z, \pi_Z, k)$,

i.e. i and p have sections.

<u>Claim</u>: There is an exact structure on PM_{equiv} that is a 'pullback' of the trivial exact structure on PM_Ω along the canonical $PM_{equiv} \longrightarrow PM_\Omega$.

Need to check:

$$\begin{tikzcd}
0 \arrow[r] & (M, \gamma_M) \arrow[r, "i"] \arrow[d, "\alpha"] & (N, \gamma_N) \arrow[r, "p"] \arrow[d, dotted] & (K, \gamma_K) \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & (T, \gamma_T) \arrow[r, dotted] & (N_T, \gamma_{N_T}) \arrow[r, dotted] & (K, \gamma_K) \arrow[r] & 0
\end{tikzcd}$$

Let's assume that M, N, K, T are invariant motives,

then (non-canonically) $N \simeq M \oplus K$ and $N_T \simeq T \oplus K$

we have to construct $\gamma_{N_T} =: \gamma$ on $T \oplus K$

from the given data:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\alpha"] & N \arrow[r, "p"] \arrow[d] \arrow[l, dashed, bend right, "s"'] & K \arrow[r] \arrow[d, "id"] \arrow[l, dashed, bend right, "\sigma"'] & 0 \\ 0 \arrow[r] & T \arrow[r] & (T\oplus K, \gamma) \arrow[r, "\pi_K"] & K \arrow[r] & 0 \end{tikzcd}$$

$s: N \to M$ in PM_Ω but not invariant

$s \circ i = id_M$

$$N \xrightarrow{(s,p)} T \oplus K$$

$$\begin{tikzcd} F_1 N \arrow[r, "(F_1\alpha \circ F_1 s, F_1 p)"] \arrow[d, "id"] & F_1 T \oplus F_1 K \arrow[r] \arrow[d, "\gamma"] & F_1 K \arrow[d, "id"] \\ F_2 N \arrow[r, "(F_2\alpha \circ F_2 s, F_2 p)"] & F_2 T \oplus F_2 K \arrow[r] & F_2 K \end{tikzcd}$$

$$F_1 T \xrightarrow{id} F_2 T$$

$$\begin{tikzcd} F_1 K \arrow[r, "?"] \arrow[dr] & F_2 T \\ & F_2 M \arrow[u, "F_2\alpha"'] \end{tikzcd}$$

$$\begin{tikzcd} F_1 K \arrow[r, "F_1 \sigma"] \arrow[d, dash] & F_1 N \arrow[d, dash] \\ F_2 K \arrow[r, "F_2 \sigma"] & F_2 N \end{tikzcd}$$

$F_2\sigma - F_1\sigma : F_1 K \to F_2 M$ $\rightsquigarrow$

Have to check that γ satisfies the cocycle condition...

In fact, one can "compute" Ext groups in $\mathrm{PM}_{\mathrm{quiv}}$:

$$\mathrm{Ext}^1\big((C,\gamma_C),(K,\gamma_K)\big) \cong \frac{\left\{ F_1 C \xrightarrow{\delta} F_2 K \;\middle|\; G_{13}\delta = G_{23}\gamma_K \circ G_{12}\delta + G_{23}\delta \circ G_{12}\gamma_C \right\}}{\left\{ d\varphi = \gamma_K \circ F_1\varphi - F_2\varphi \circ \gamma_C \text{ where } \varphi : C \to K \right\}}$$

And for $\mathrm{Ext}^{>n}$ there are also formulae ...

<u>How to get examples of extensions?</u>

<u>Lemma.</u> If $f : M_{\mathrm{inv}}(X) \longrightarrow M_{\mathrm{inv}}(Y)$ splits, then there is a "cokernel", i.e.

$$0 \to M_{\mathrm{inv}}(X) \longrightarrow M_{\mathrm{inv}}(Y) \longrightarrow \underbrace{(N,\gamma_N)}_{\text{not neccesarily invariant}} \to 0$$

Similarly, for kernels:

if $\quad M_{\mathrm{inv}}(Y) \xrightarrow{p} M_{\mathrm{inv}}(Z)$ splits,

then ...

- You can use this to construct $M_{inv}(\mathbb{P}^d)$ out of "1-dimensional" motives:

$$0 \to M_{inv}(\mathbb{P}^{d-1}) \hookrightarrow M_{inv}(\mathbb{P}^d) \longrightarrow (\mathbb{Z}(d), f) \to 0$$

and so on.

- Similarly, if $0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_d = \mathcal{E}$ is a filtration on a vector bundle $\mathcal{E}$ on X

we get that $M(\mathbb{P}_X(\mathcal{E}))$ has an admissible filtration by $M(\mathbb{P}_X(\mathcal{E}_i))$.

- $$\begin{tikzcd} E \arrow[r] \arrow[d] & Bl_Z X \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

$E \simeq \mathbb{P}_Z(N)$, we construct

$$0 \to \widetilde{M}_{inv}(E) \to M_{inv}(E) \to M_{inv}(Z) \to 0$$

and $\underbrace{0 \to \widetilde{M}_{inv}(E) \to M_{inv}(Bl_Z X) \to M_{inv}(X) \to 0}$

<u>Conjecture</u>: All Chow decompositions of motives lift to admissible filtrations in PMequiv.

Realization functor on PM_{equiv}:

$$Re : PM_{equiv} \longrightarrow (\mathbb{L}, \mathbb{LB})\text{-comod}$$

$$(M, \varphi) \longmapsto \text{"descent data" on } \Omega^*(M)$$

$$(\mathbb{L}, \mathbb{LB})\text{-comod} \simeq QCoh(\mathcal{M}_{fg}) :=$$

$$:= 2\text{-Lim}\left(QCoh(\operatorname{Spec}\mathbb{L}) \rightrightarrows QCoh(\operatorname{Spec}\mathbb{LB}) \Rrightarrow QCoh(\operatorname{Spec}\mathbb{LB}\times B')\right)$$

claim: Re is an exact functor. (easy)

Might be interesting

$$Re : Ext^i(\mathbb{Z}(k), \mathbb{Z}(m)) \longrightarrow$$

$$\longrightarrow \underbrace{Ext^i_{(\mathbb{L},\mathbb{LB})}(\underset{\mathbb{L}(k)}{\mathbb{L}z^k}, \underset{\mathbb{L}(m)}{\mathbb{L}z^m})}_{\text{related to Adams-Novikov spectral sequence}}$$