Hermitian K-theory of Dedekind domains

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§1 Dramatis personae

Let R be a commutative ring, a <u>symmetric bilinear form</u> is a pair (P, b) where P is a fg projective R-module & $b : P \otimes_R P \to R$ is <u>symmetric</u> $b(x,y) = b(y,x)$
$\left(b \in \mathrm{Hom}_R(P \otimes_R P, R)^{C_2} \right)$.

We say that b is <u>unimodular</u> if $P \xrightarrow[\sim]{\tilde{b}} \mathrm{Hom}_R(P,R)$ is an isomorphism

<u>Q</u>: We want to classify unimodular symmetric bilinear forms.

<u>Ex</u>: · $R = F = \bar{F}$ alg. closed field, all $^{un.}$ s.b. forms of the same rank are isomorphic

· $R = \mathbb{R}$ unim. symm. bil forms are classified by the rank and the signature

· $R = \mathbb{Z}$ there are a <u>lot</u> of unimod. symm. bilinear forms.

<u>Remark</u>: You can take the sum of two symm. bil. forms $(P,b) \oplus (P',b') = (P \oplus P', b \oplus b')$

$(b \oplus b')(x,x';y,y') = b(x,y) + b(x',y')$.

$\Rightarrow$ isoclasses of (un.) symm. bilinear forms form a $\overset{\text{comm.}}{\text{monoid}}$ $\Rightarrow$ we can group complete

<u>Def</u>: $GW_0^{gs}(R) = \{\text{isoclasses of unim. symm. bil forms}\}^{gp}$ (Grothendieck-Witt group)

<u>Ex</u>: $GW_0^{gs}(\mathbb{Z}) \xrightarrow{\sim} GW_0^{gs}(\mathbb{R})$

<u>Def</u>: $GW_{cl}^{gs}(R) \simeq \{\text{gpd of unimod symm bilinear forms}\}^{gp}$ (Groth-Witt <u>space</u>)

(Karoubi - Villamayor '71)

<u>Ex</u>: $GW_{cl}^{gs}(\mathbb{Z}) = GW_0^{gs}(\mathbb{Z}) \times BO_{\infty,\infty}(\mathbb{Z})^{+}$ $\quad \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ 0 & & & & \ddots \end{pmatrix}$

<u>Remark</u>: We can generalise: we another interesting notions: a <u>quadratic form</u> (P,q)
P fg proj R-mod, $q : P \to R$ function s.t.

① $q(rx) = r^2 q(x)$ ② $b_q(x,y) = q(x+y) - q(x) - q(y)$ is bilinear (<u>polarisation</u>)

Remark: We can generalise: we another interesting notions: a quadratic form (P, q)

P f.g. proj R-mod, $q : P \to R$ function s.t.

① $q(rx) = r^2 q(x)$ ② $b_q(x,y) := q(x+y) - q(x) - q(y)$ is bilinear (polarisation)

We say q is unimodular if b_q is.

Concretely q is a function given by a degree 2 homogeneous polyn. if P is free.

Ex: $P = \mathbb{Z}^{\oplus 2}$ $\quad q(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

b_q is the form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Remark: $q : P \to R$ is quadratic iff $\exists\, \hat{b} : P \otimes_R P \to R$ (not nec sym.!) s.t. $q(x) = \hat{b}(x,x)$.

In fact the abelian group of q. forms is isomorphic to $\mathrm{Hom}_R(P \otimes_R P, R)/C_2$ & the polaris. is the image under the norm map

$$\mathrm{Hom}_R(P \otimes_R P, R)/C_2 \longrightarrow \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$$
$$[\hat{b}] \longmapsto \hat{b}(x,y) + \hat{b}(y,x).$$

You can do the same thing for q. forms & you get $GW_d^{gq}(R)$, and a polaris. map

polarisation: $GW_d^{gq}(R) \to GW_d^{gs}(R)$

Def: Let R be a ring. Then a Poincaré structure on R is a functor

$$Ϙ : \mathrm{Proj}_R^{op} \to \mathrm{Sp} \qquad (\mathrm{Proj}_R = \text{cat of f.g. proj left } R\text{-modules})$$

s.t. $\exists\, D : \mathrm{Proj}_R^{op} \to \mathrm{Proj}_R$ duality and $n \in \mathbb{Z}$ s.t. $\forall P, Q \in \mathrm{Proj}_R$

$$Ϙ(P \oplus Q) \simeq Ϙ(P) \oplus Ϙ(Q) \oplus \Sigma^n \mathrm{Hom}_R(P, DQ)$$

Remark: A duality is a functor $D : \mathrm{Proj}_R^{op} \to \mathrm{Proj}_R$ + a map $\eta : 1 \xrightarrow{\sim} D^{op} D$ s.t.

$$\begin{tikzcd} D^{op} \arrow[r, "\eta D^{op}"] \arrow[dr, equal] & D^{op} D D^{op} \arrow[d, "D^{op}\eta"] \\ & D^{op} \end{tikzcd}$$

Def: Let R be a ring. Then a <u>Poincaré structure</u> on R is a functor

$$Ϙ : \mathrm{Proj}_R^{op} \to \mathrm{Sp} \qquad (\mathrm{Proj}_R = \text{cat of f.g. proj left } R\text{-modules})$$

s.t. $\exists\, \mathbb{D} : \mathrm{Proj}_R^{op} \to \mathrm{Proj}_R$ duality and $n \in \mathbb{Z}$ s.t. $\forall P, Q \in \mathrm{Proj}_R$

$$Ϙ(P \oplus Q) \simeq Ϙ(P) \oplus Ϙ(Q) \oplus \Sigma^n \mathrm{Hom}_R(P, \mathbb{D}Q)$$

$Ϙ(P) =$ "group of q. forms on P" $\qquad Ϙ(P) \oplus Ϙ(Q) \hookrightarrow Ϙ(P \oplus Q)$

$\mathrm{Ab} \subseteq \mathrm{Sp}$ as Eilenberg-MacLane spectra

Ex:
- $Ϙ^{s}(P) = \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$
  $(n = 0,\ \mathbb{D}P = \mathrm{Hom}_R(P, R))$
- $Ϙ^{q}(P) = \mathrm{Hom}_R(P \otimes_R P, R)_{C_2}$ $\qquad (n=0,\ \mathbb{D}P = \mathrm{Hom}_R(P,R))$
- $Ϙ^{s}_{-}(P) = \mathrm{Hom}_R(P \otimes_R P, R)^{-C_2}$ $\qquad (h(x,y) \mapsto -h(y,x))$
  $(n = 0,\ \mathbb{D}P = \mathrm{Hom}_R(P,R)$ but the double dual id. changes$)$
- L line bundle $\quad Ϙ^{s}_L(P) = \mathrm{Hom}_R(P \otimes P, L)^{C_2}$
- R ring w/ involution (e.g. $R = \mathbb{Z}[G]$, $\sigma(\sum a_g g) = \sum a_g g^{-1}$)
  $$Ϙ(P) = \mathrm{Hom}_{R \otimes_{\mathbb{Z}} R}(P \otimes_{\mathbb{Z}} P, R)^{C_2}$$
  ("hermitian forms": $h(x,y) = \overline{h(y,x)}$, $h(x, zy) = \bar{z}\, h(x,y)$, ...)

Def: R ring, an R-module w/ involution M is an $(R \otimes R)$-module M, equipped w/ $\sigma : M \to M$ s.t.

1. $\sigma^2 = 1$
2. $\sigma((z \otimes y) m) = (y \otimes z)\, \sigma(m)$

Ex:
- R comm. every R-module is an R-mod. w/ involution
- R ring w/ involution ($\mathbb{Z}[G]$) $\qquad R$ has a canonical R-module w/ inv. structure

$(z \otimes y)x = z x \sigma(y) \qquad \sigma$ is the inv.

Def.: R ring, an R-module w/ involution M is an $(R\otimes R)$-module M, equipped w/ $\sigma: M\to M$ s.t.

① $\sigma^2 = 1$

② $\sigma((r\otimes s)m) = (s\otimes r)\sigma(m)$

Ex.: · R comm. every R-module is an R-mod. w/ involution

· R ring w/ involution ($\mathbb{Z}[G]$) R has a canonical R-module w/ inv. structure

$(r\otimes s)x = r x \sigma(s)$ σ is the inv.

M is <u>invertible</u> if it is proj as an R-module, $R \xrightarrow{\sim} \mathrm{End}_R(M)$.

<u>Construction:</u> M R-module w/ involution, in particular it is an abelian group w/ a C_2-action

$\Rightarrow \exists\, M^{tC_2}$ Tate const. (spectrum s.t. $\pi_* M^{tC_2} = \hat{H}^{-*}(C_2; M)$), and this comes eq. w/ an R-module (i.e. an element of $\mathcal{D}(R)$)

M^{tC_2} is an $(R\otimes R)^{tC_2}$-module & we use $R \to (R\otimes R)^{tC_2}$ Tate diagonal

not $\mathbb{Z}$-linear

<u>Theorem:</u> Let R be a ring, then Poincaré structures on R are determined by an invertible R-module w/ involution M, $n\in\mathbb{Z}$, and a map $\Sigma^{-n}X \xrightarrow{\alpha} M^{tC_2}$ in $\mathcal{D}(R)$, $X \in \mathcal{D}(R)$

$$\begin{array}{ccc} Ϙ^\alpha_M(P) & \longrightarrow & \mathrm{Hom}_R(P,X) \\ \downarrow & & \downarrow \alpha_* \\ \Sigma^n \mathrm{Hom}_R(P\otimes P, M)^{hC_2} & \longrightarrow & \Sigma^n \mathrm{Hom}_R(P\otimes P, M)^{tC_2} \simeq \Sigma^n \mathrm{Hom}_R(P, M^{tC_2}) \end{array}$$

fact: the lhs is exact

Ex.: · R comm. you can take $M = R$, $X = \tau_{\geq 0} R^{tC_2} \xrightarrow{\alpha} \underbrace{R^{tC_2}}$, $n=0$

$\Rightarrow Ϙ^{gs} = Ϙ^\alpha_R$ (exercise)

· $M = R$ $X = \tau_{\geq 2} R^{tC_2} \xrightarrow{\alpha} R^{tC_2}$ $n=0$

$Ϙ^{gq} = Ϙ^\alpha_R$ (exercise)

$\Rightarrow Ϙ^{\geq n} := Ϙ^{\alpha_n}_R$ $\alpha_n: \tau_{\geq n} R^{tC_2} \to R^{tC_2}$.

$Ϙ^{\geq 0} = Ϙ^{gs}$, $Ϙ^{\geq 2} = Ϙ^{gq}$, $(Ϙ^{\geq 1} = Ϙ^{ge})$

§2 GW & L

$\mathfrak{Q} : \mathrm{Proj}_R^{\omega, op} \to Sp$ can be extended to $\mathcal{D}^p(R)^{op} \to Sp$ (non abelian derived functor à la Dold-Puppe)

Concretely the square

$$\begin{tikzcd}
\mathfrak{Q}^{\alpha}_M(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_R(P,X) \arrow[d, "\alpha_*"] \\
\Sigma^n \mathrm{Hom}_R(P\otimes P, M)^{hC_2} \arrow[r] & \Sigma^n \mathrm{Hom}_R(P\otimes P, M)^{tC_2} \simeq \Sigma^n \mathrm{Hom}_R(P, M^{tC_2})
\end{tikzcd}$$

$\hat{\mathfrak{Q}}$

is still a pullback $\forall P \in \mathcal{D}^p(R)$

Def: A Poincaré object is $(P \in \mathcal{D}^p(R), q \in \Omega^\infty \mathfrak{Q}(P))$ s.t. the image

$$q \in \Omega^\infty \mathfrak{Q}(P) \to \Omega^\infty \mathfrak{Q}(P\oplus P) \simeq \Omega^\infty \mathfrak{Q}(P) \times \Omega^\infty \mathfrak{Q}(P) \times \Omega^\infty \mathrm{map}(P, \Sigma^n DP) \to \mathrm{Map}(P, \Sigma^n DP)$$

$P \xrightarrow{\simeq} \Sigma^n DP$ is an equivalence.

Def: A Lagrangian on a Poincaré object (P,q) is a pair $(L \xrightarrow{f} P,\ \gamma : f^* q \sim 0)$ s.t. ($L \in \mathcal{D}^p(R)$) ($L \subseteq L^\perp$)

$$L \xrightarrow{f} P \simeq \Sigma^n DP \xrightarrow{Df} \Sigma^n DL$$

is a fiber sequence $(L = L^\perp)$

Ex: · (P, b) unimodular symm bil form $\Rightarrow$ is a Poincaré object in $(R, \mathfrak{Q}^{gs})$

· $L \xrightarrow{f} P$ Lag. in degree 0 $(L \in \mathrm{Proj}_R^\omega)$

$f^* b = b \circ (f \otimes f) \in \Omega^\infty \mathfrak{Q}^{gs}(L)$ there's only one possible homotopy

$\Rightarrow$ asking $f^* b = 0$ $\left(b|_{L\otimes L} = 0 \right)$

$$L \xrightarrow{f} P \simeq DP \xrightarrow{Df} DL$$

is a short exact seq. Therefore f is injective $\Rightarrow L \subseteq P$

Moreover $\underbrace{\ker Df}_{=: L^\perp} = \{x \in P \mid b(x,y) = 0\ \forall y \in L\}$ because Df sends x to $b(x,-)|_L$

A Lagrangian in dimension 0 is a subspace $L \subseteq P$ s.t. $L = L^\perp$.

Ex: $(P, h) = \left(\mathbb{R}^{\oplus 2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

$L = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is a Lagrangian

Ex: $P \in \mathrm{Perf}_{\mathbb{R}}$ $\quad$ $\mathrm{Hyp}\, P = \left(\overset{0}{P} \oplus \overset{-1}{DP}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ $[(x, \varphi; y, \psi) \mapsto \psi x + \varphi y]$

Hyp P is a unim. sym. bil. form. w/ Lagrangian $P \oplus 0 \subseteq P \oplus DP$

Ex: $L = [\overset{0}{N} \to \overset{-1}{Q}]$

$$\begin{array}{ccccc} N & \to & Q & \to & \cdots \\ \downarrow f & & \downarrow & & \\ P & \to & 0 & \to & \cdots \end{array}$$

f is just a map $N \to P$

The condition that f is a Lagrangian is the same as $N \to P \oplus \mathrm{Hyp}\, Q$ is injective & a Lagrangian there.

Def: $L_n(R, \mathfrak{Q}) := \{$isoclasses of $\overset{\text{cob. manifold}}{\text{Poinc.}}$ objects for $\Sigma^n \mathfrak{Q}\}$ $\Big/ \{$Poinc. objects that admit a Lagrangian$\}$

Ex: $L_0(R, \mathfrak{Q}^{gs}) = W^s R = \{$unimod. sym. bilinear forms$\} \Big/ \{$those that admit a Lagrangian subspace$\}$

$W^s R \to L_0(R, \mathfrak{Q}^{gs})$

Let me show that it is surjective.

$(P \in \mathcal{D}^p R, \ q \in \Omega^\infty \mathfrak{Q}(P))$ $\qquad$ $n > 0$

$$\begin{array}{ccccccc} 0 \leftarrow & P_{-n} & \leftarrow & 0 & \leftarrow 0 & \cdots \\ & \downarrow & & \downarrow & & \\ \text{Suppose } P \text{ is rep. by a complex: } 0 \leftarrow & P_{-n} & \leftarrow & P_{-n+1} & \leftarrow & \cdots \end{array}$$

$f : P_{-n}[-n] \to P$

Claim: This is isotropic i.e. $f^* q = 0$. In fact

$\pi_0 \mathfrak{Q}^{gs}(P_{-n}[-n]) = 0$

$\underset{2n\text{-connective}}{\simeq} \mathrm{map}(P_{-n}[-n] \otimes P_{-n}[-n], R)_{hC_2} \to \underset{1\text{-connective}}{\mathfrak{Q}^{gs}(P_{-n}[-n])} \to \mathrm{map}(P_{-n}[-n], \underset{n\text{-connective}}{\text{connective}})$

$\Rightarrow \quad P_{-n}[-n] \to P \oplus \left(P_{-n+1}[-n+1] \leftarrow \cdots \right)^{-n+1}$ Lagr.

$\Rightarrow [P, q]$ is repr. by something starting in degree $-n+1$ (surgery)

$\Rightarrow \dots \Rightarrow [P, q]$ connective, but then $DP \simeq P$ is also connective $\Rightarrow P$ proj module in degree 0. □

To a pair (R, Q) we associate 2 spectra

$\mathrm{GW}(R, Q)$ and $L(R, Q)$

w/ the following properties

① There is a cartesian square

$$\begin{tikzcd} \mathrm{GW}(R,Q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L(R,Q) \arrow[d] \\ K(R,Q)^{hC_2} \arrow[r] & K(R,Q)^{tC_2} \end{tikzcd}$$

where the C_2-action on $K(R)$ sends P to $\Sigma^4 DP$

② $\pi_n L(\mathcal{C}, Q) = L_n(\mathcal{C}, Q)$

③ $\exists$ a fiber sequence $\mathrm{GW}(R, \Sigma Q) \to K(R, Q) \to \mathrm{GW}(R, Q)$

④ (Hebestreit-Steimle): $\Omega^\infty \mathrm{GW}(R, Q^{gs}) \simeq \mathrm{GW}^{gs}_{cl}(R)$

$\Omega^\infty \mathrm{GW}(R, Q^{gq}) \simeq \mathrm{GW}^{gq}_{cl}(R)$

⑤ If we let $Q^s(P) = \mathrm{Hom}(P \otimes P, R)^{hC_2}$, $Q^q(P) = \mathrm{Hom}(P \otimes P, R)_{hC_2}$

$L(R, Q^s)$ and $L(R, Q^q)$ are 4-periodic.

⑥ $L(R, Q^q) \to L(R, Q^{\geq n})$ is an iso in degrees $\leq 2n-3$

& injective in degree $= 2n-2$

⑦ If R Noetherian ring of global dim d

$L(R, Q^{\geq n}) \to L(R, Q^s)$ is an iso in degrees $\geq d-2n+3$

& injective in degree $= d-2n+2$

$\Rightarrow$ If R Noetherian of finite global dim $\Rightarrow$ there are only finitely many L-groups to compute

Specialise for fields: $R = F$

Fields are regular Noeth of dim 0

$$\pi_* L(F, \ell^{gs}) = \begin{cases} W^s F & * \text{ div by } 4,\ * \geq 0 \\ W^q F & * \text{ div by } 4 \quad * \leq -4 \\ \mathrm{Im}(W^q F \to W^s F) & * = -2 \\ 0 & \text{otherwise} \end{cases}$$

(2 if char $F = 2$)

Now let $F = \mathbb{F}_q$ finite field of characteristic 2

$$\pi_* L(F, \ell^s) = \mathbb{F}_2[u^{\pm 1}] \qquad |u| = 2$$

$K(F, \ell^s)^{tC_2}$. By Quillen we know $\pi_* K(F)$ are odd torsion groups in dim $\neq 0$

$$\Rightarrow K(F,\ell)^{tC_2} \simeq K_0(F,\ell)^{tC_2} = \mathbb{Z}^{tC_2} \Rightarrow \pi_* K(F,\ell)^{tC_2} = \mathbb{F}_2[u^{\pm 1}]$$

$\Rightarrow \pi_* L(F, \ell^s) \to \pi_* K(F, \ell^s)^{tC_2}$ is a graded ring map $\mathbb{F}_2[u^{\pm 1}] \circlearrowleft$

$\Rightarrow$ it is an iso.

$$\begin{tikzcd} GW(F, \ell^s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L(F, \ell^s) \arrow[d] \\ K(F, \ell^s)^{hC_2} \arrow[r] & K(F, \ell^s)^{tC_2} \end{tikzcd}$$

$$\Rightarrow (\text{HFPSS}) \quad \pi_* GW(F, \ell^s) = \begin{cases} \pi_* K(F, \ell^s) & * \equiv 3\ (4),\ * > 0 \\ GW_0 & * = 0 \\ \mathbb{F}_2 & * \text{ even},\ * < 0 \\ 0 & \text{otherwise} \end{cases}$$

But $GW(F, \ell^{gs}) \to GW(F, \ell^s)$ is an iso in degrees ≥ 0.

$\Rightarrow$ We found the homotopy groups of the GW space of F.

Dedekind domains

R Ded dom of dim 1

$L(R, \mathcal{Q}^{gs}) \to L(R, \mathcal{Q}^{s})$ is an iso in degrees ≥ 0

$\Rightarrow GW(R, \mathcal{Q}^{gs}) \to GW(R, \mathcal{Q}^{s})$ is an iso

Thm (Localisation sequence): Let S a set of primes w/ chosen uniformisers $\Rightarrow \exists$ a fiber seq.

$$\bigoplus_{p \in S} L(R/p, [\mathcal{Q}^{s}]) \to L(R, \mathcal{Q}^{s}) \to L(\underbrace{R[S^{-1}]}_{\subseteq \operatorname{Frac}(R)}, \mathcal{Q}^{s})$$

$$\Rightarrow L(R, \mathcal{Q}^{s}) \to L(R[S^{-1}], \mathcal{Q}^{s}) \xrightarrow{\partial} \bigoplus_{p \in S} L(R/p, \mathcal{Q}^{s})$$

Lemma: $\partial_0 : \pi_0 L(R[S^{-1}]) \to \pi_0 L(R/p) \qquad (\psi')$

$$\overset{\shortparallel}{W}(R[S^{-1}]) \to W(R/p)$$

sends $\langle u \rangle \mapsto 0$ where $u \in R \times_{R/p} (R/p)^{\times}$

$\langle u\pi \rangle \mapsto \langle \bar{u} \rangle$ $\quad \pi$ is the uniformiser

Thm:

$\dim R \neq 2$

$$\pi_n L(R, \mathcal{Q}^{s}) = \begin{cases} W^s R & n \equiv 0 \ (4) \\ 0 & n \equiv 2 \\ \bigoplus_{p|2} W^s R/p & n \equiv 1 \\ \operatorname{coker}\underbrace{\left(\partial_0 : W^s(\operatorname{Frac} R) \to \bigoplus_p W^s(R/p)\right)}_{\operatorname{Pic} R/2 \text{ if } \operatorname{Frac}(R) \text{ global field}} & n \equiv 3 \end{cases}$$

Proof: $n \equiv 0$

Let's write the les for S = set of all primes

$$\pi_* L(\underset{\overset{\shortparallel}{F}}{R[S^{-1}]}) = \begin{cases} W^s R[S^{-1}] & * \equiv 4 \\ 0 & \text{otherwise} \end{cases} \qquad \Rightarrow n \equiv 2$$

$$0 \to 0 \to W^s R \to W^s F \xrightarrow{\partial_0} \bigoplus_p W^s(R/p) \to \boxed{\pi_{-1} L(R)} \to \underset{\overset{\shortparallel}{0}}{\pi_{-1} L(F)} \qquad n \equiv 3$$

$\pi_* \ldots$ $\;\; \parallel \;\; F \qquad \{ 0 \;\; \text{otherwise} \qquad\qquad n=$

$$\bigoplus_p \pi_1 L(R/p) \to W^s R \to W^s F \xrightarrow{\partial_0} \bigoplus_p W^s(R/p) \to \boxed{\pi_{-1} L(R)} \to \pi_{-1} L(F)$$

$\pi_1 L(R/p) = 0$, $\quad W^s R = \pi_0 L(R)$

$$\pi_1 L(F) \leftarrow \pi_1 L(R) \leftarrow \bigoplus_p \underbrace{\pi_2 L(R/p)}_{} \leftarrow \pi_2 L(F)$$

$\pi_1 L(F) = 0$, $\quad \pi_2 L(F) = 0$

$\pi_2 L(R/p) = W^s R/p$ if $2|p$

$= 0$ oth.

$$\Rightarrow \pi_1 L(R) = \bigoplus_{p|2} W^s(R/p) \quad . \square$$

- $\pi_* L^{gs}(R) = \begin{cases} W^s R & * \equiv 0 \;(2) \\ Pic/2 & * \equiv 1 \;(2) \end{cases}$ $\quad * \geq 0 \qquad$ if Frac R has char 2

- $L^{gq}_{q \geq 2}(R) \simeq \Sigma^4 L^{gs}(R)_{q \geq 0}$ (Karoubi periodicity: R^{tC_2} is 2-periodic, $\tau_{\geq 2} R^{tC_2} = \Sigma^2 \tau_{\geq 0} R^{tC_2}$)

$$\pi_* L^q(R) = \begin{cases} W^q(R) & * \equiv 0 \;(4) \\ 0 & * \equiv 1 \;(4) \\ \bigoplus_{p|2} W^q(R) & * \equiv 2 \;(4) \\ \sim & * \equiv 3 \;(4) \end{cases}$$

char Frac $R \neq 2$

For any com. ring $\quad L^{gs}_-(R) \simeq \Sigma^2 L^{gs}_+(R) \quad$ (yet another inst of Karoubi periodicity)

$$L(R, \mathcal{Q}^{\geq n}_L) \simeq \Sigma^2 L(R, \mathcal{Q}^{\geq n+1}_{-L})$$

<u>pre Theorem</u> (Calmès–Harpaz–N): $X^{qcqs} \mapsto GW(X, \mathcal{Q}^{\geq n}_L)$ is a Nisnevich sheaf $\forall n$.

but it's $\mathbb{A}^1$-invariant only when $n = -\infty$ (i.e. $\mathcal{Q}^s$)

W^q is not $\mathbb{A}^1$-inv $\quad [x^2 + xy + ty^2] \in W^q(\mathbb{F}_2[t])$ $\xrightarrow{t \mapsto 0} 0 \in W^q(\mathbb{F}_2) \simeq \mathbb{Z}/2$

$\xrightarrow{t \mapsto 1} 1 \in W^q(\mathbb{F}_2)$

$q(x,y) = x^2 + xy + ty^2$ q. fm over $\mathbb{F}_2[t]$

its pol. is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathrm{Arf}(q) = t$

$$(\mathcal{C}', \mathcal{Q}') \to (\mathcal{C}, \mathcal{Q}) \to (\mathcal{C}'', \mathcal{Q}'')$$

- $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ (split) Verdier seq.
- $\mathcal{Q}' = \mathcal{Q}|_{\mathcal{C}'}$
- $\mathcal{Q}''$ is the left Kan ext of $\mathcal{Q}$ to $\mathcal{C}''$