Свойства комплекса Роста, II, 10.11.20

План: 49.E 50 граничн. отобр-е

X – кон типа$/k$

$$\begin{tikzcd} Z \arrow[r, hook, "i"'] & X & U = X\setminus Z \arrow[l, hook', "j"] \end{tikzcd}$$

i конечное, j плоское отн. разм 0

$X_{(p)} = U_{(p)} \sqcup Z_{(p)}$ $\forall p \geq 0$

$C_p(X) = C_p(Z) \oplus C_p(U)$

$$\begin{tikzcd} 0 \arrow[r] & C_*Z \arrow[r, "i_*"'] & C_*X \arrow[r, "j^*"'] \arrow[l, dashed, bend right, "\omega"'] & C_*U \arrow[r] \arrow[l, dashed, bend right, "v"'] & 0 \end{tikzcd}$$

$\partial^U_Z = \omega d_X v : C_p(U) \to C_{p-1}(Z)$

Пример: $X = \mathbb{A}^1$ $Z = \{0\}$ $U = \mathbb{A}^1 \setminus 0$

$\partial^U_Z(\{t\}[U]) = [Z]$

49.33 $a \in k[X]^*$ $a' = a/_U$ $a'' = a/_Z$

$\forall \alpha \in C_*(U)$ $\partial^U_Z(\alpha\{a'\}) = \partial^U_Z(\alpha)\{a''\}$

$\partial^U_Z(\{a'\}\alpha) = -\{a''\}\partial^U_Z(\alpha)$

$$\begin{cases} \partial_v(\alpha\{a\}) = \partial_v(\alpha)\{\bar a\} \\ N_{E_i/E}(r_{E_i/E}(\alpha)\beta) = \alpha N_{E_i/E}(\beta) \end{cases}$$

$\partial^X_{X'} = \sum N_{E_i/E}\, \partial_{v_i}$

$u \in U$

$$\begin{tikzcd} & U' \arrow[r, hook] & X' \arrow[d] \\ U'_u \arrow[ur] \arrow[r, "\cong"] \arrow[d] & X'_{j(u)} \arrow[ur] \arrow[d] & \\ & U \arrow[r, hook] & X \\ \operatorname{Spec} k(u) \arrow[ur] \arrow[r, "\cong"] & \operatorname{Spec} k(j(u)) \arrow[ur] & \end{tikzcd}$$

49.36

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, "g"'] & X' \arrow[d, "f"] & U' \arrow[l, hook'] \arrow[d, "h"] \\ Z \arrow[r, hook] & X & U \arrow[l, hook'] \end{tikzcd} \qquad U' = X'\setminus Z' \qquad U = X \setminus Z$$

i) f, g, h – собств

$$\begin{tikzcd} C_p(U') \arrow[r, "\partial^{U'}_{Z'}"] \arrow[d, "h_*"'] & C_{p-1}(Z') \arrow[d, "g_*"] \\ C_p(U) \arrow[r, "\partial^U_Z"] & C_{p-1}(Z) \end{tikzcd}$$

ii) $\exists$ оба квадрата пулбэки; f – плоский отн. разм d

$$\begin{tikzcd} C_p(U) \arrow[r, "\partial^U_Z"] \arrow[d, "h^*"'] & C_{p-1}(Z) \arrow[d, "g^*"] \\ C_{p+d}(U') \arrow[r, "\partial^{U'}_{Z'}"] & C_{p+d-1}(Z') \end{tikzcd}$$

Д-во

$$\begin{tikzcd} C_p(U') \arrow[r, "v'"'] \arrow[d, "h_*"] & C_p(X') \arrow[r, "d_{X'}"'] \arrow[d, "f_*"] & C_{p-1}(X') \arrow[r, "\omega'"'] \arrow[d, "f_*"] & C_{p-1}(Z') \arrow[d, "g_*"] \\ C_p(U) \arrow[r, "v"'] & C_p(X) \arrow[r, "d_X"'] & C_{p-1}(X) \arrow[r, "\omega"'] & C_{p-1}(Z) \end{tikzcd}$$

$x' \notin Z' \to f(x') \notin Z$

$$\begin{tikzcd} C_p(U) \arrow[r, "v"'] \arrow[d, "h^*"] & C_p(X) \arrow[r, "d_X"'] \arrow[d, "f^*"] & C_p(X) \arrow[r, "\omega"'] \arrow[d, "f^*"] & C_p(Z) \arrow[d, "g^*"] \\ C_{p+d}(U') \arrow[r, "v'"'] & C_{p+d}(X') \arrow[r, "d_{X'}"'] & C_{p+d}(X') \arrow[r, "\omega'"'] & C_{p+d}(Z') \end{tikzcd}$$

$U'_u \cong X'_{j(u)}$ $X'_{i(z)} \cong Z'_z$

$Z_1, Z_2 \hookrightarrow X$

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, "f"] & Z_2 \arrow[d, "f"] & T_2 \arrow[l, hook'] \arrow[d, "f"] \\
Z_1 \arrow[r, hook] & X & U_1 \arrow[l, hook'] \\
T_1 \arrow[r, hook] \arrow[u, "\varphi"] & U_2 \arrow[u, "\varphi"] & U \arrow[l, hook'] \arrow[u, "\varphi"]
\end{tikzcd}$$

$T_1 = Z_1 \setminus Z_2 \quad T_2 = Z_2 \setminus Z_1$

$U_i = X \setminus Z_i \quad U = U_1 \cap U_2$

$Z = Z_1 \cap Z_2$

$\partial_t = \partial^{T_2}_{Z} \quad \partial_l = \partial^{T_1}_{Z}$

$\partial_b = \partial^{U}_{T_1} \quad \partial_r = \partial^{U}_{T_2}$

49.37: $\partial_l \partial_b + \partial_t \partial_r : C_*(U) \to C_{*-2}(Z)$ гомотопен 0 морф. компл.

Д-во $C_*(X) = C_*(U) \oplus C_*(T_1) \oplus C_*(T_2) \oplus C_*(Z)$

$$d_X = \begin{pmatrix} d_U & * & * & * \\ \partial_b & * & * & * \\ \partial_r & * & * & * \\ h & \partial_l & \partial_t & d_Z \end{pmatrix}$$

$d_X^2 = 0 \qquad h d_U + d_Z h + \partial_l \partial_b + \partial_t \partial_r = 0$

d_X^2

50 — Внешнее произведение

X, Y - отд. кон. типа схема $/F$

$k(x) \to F(x) \qquad C_p(X) \times C_q(Y) \to C_{p+q}(X \times Y)$

$(\alpha, \beta) \longmapsto$ "$\alpha \times \beta$"

Утв-е $x_0 \in X_{(p)}, y_0 \in Y_{(q)}$

точки $\mathrm{Spec}(F(x_0) \otimes F(y_0)) \leftrightarrow \{$точки $v \in X \times Y$, $pr_X v = x_0$, $pr_Y v = y_0\}$

$\boxed{\swarrow}$

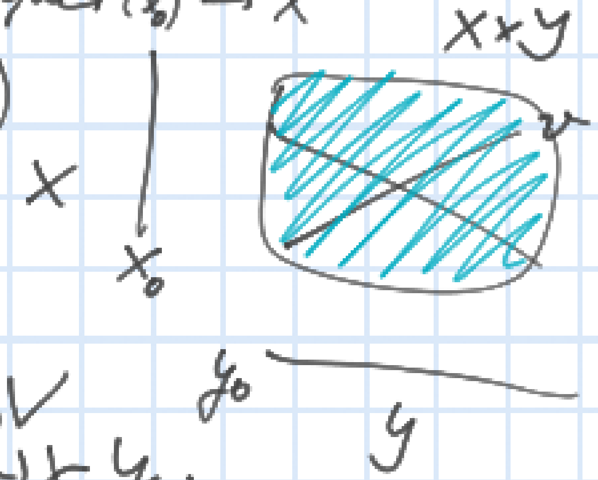
$$\begin{tikzcd}
 & \mathrm{Spec}\, F(x_0) \arrow[r] & X \\
\mathrm{Spec}\, F(v) \arrow[ur] \arrow[r, "v"] \arrow[dr] & X \times Y \arrow[ur] \arrow[dr] & \\
 & \mathrm{Spec}\, F(y_0) \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(x_0) \arrow[dl] \arrow[dr] & \\
F(v) & & F(x_0) \otimes F(y_0) \arrow[ll] \\
 & F(y_0) \arrow[ul] \arrow[ur] & p_v \arrow[u]
\end{tikzcd}$$

$\boxed{\nearrow}$

$$\begin{tikzcd}
 & & \mathrm{Spec}\, F(y_0) \arrow[r] & Y \\
\mathrm{Spec}\, F(p) \arrow[r] & \mathrm{Spec}(F(x_0) \otimes F(y_0)) \arrow[ur] \arrow[r] \arrow[dr] & X \times Y \arrow[ur] \arrow[dr] & \\
 & & \mathrm{Spec}\, F(x_0) \arrow[r] & X
\end{tikzcd}$$

Утв-е Общие точки $\mathrm{Spec}(F(x)) \times \mathrm{Spec}(F(y))$ соотв. $v \in X \times Y$

$\dim(v) = \dim(x_0) + \dim(y_0)$



$$(\alpha \times \beta)_v = \begin{cases} 0 : pr_X(v) \notin X_{(p)} \vee pr_Y(v) \notin Y_{(q)} \\ l_v \cdot r_{F(v)/F(x)}(\alpha_x) \cdot r_{F(v)/F(y)}(\beta_y) \end{cases}$$

$v \in (X \times Y)_{p+q}$

$x_{(p)} \quad y_{(q)}$

$l_v = l(\mathcal{O}_{\mathrm{Spec}\, F(x) \otimes F(y);\, p_v})$

1°) $\alpha\in C_{p,n}X$ $\quad :\beta\in C_{q,m}(Y)$

$\bigoplus_{x\in X_{(p)}} K_{p+n}(F(x))$

$$\beta\times\alpha=(-1)^{(p+n)(q+m)}(\alpha\times\beta)$$

2°) $x\in X$ $\quad Y_x = Y\times \operatorname{Spec}F(x)$

пов. разм-ть нар

плоск отн разм

$$\begin{tikzcd} Y_x \arrow[r, "i_x"] \arrow[d] \arrow[rr, bend left, "h_x"] & X\times Y \arrow[r, "pr_Y"] \arrow[d] & Y \\ \operatorname{Spec}F(x) \arrow[r] & X & \end{tikzcd}$$

Y_x - схема $/F(x)$

$C_*(Y_x)$ модуль $/K_*(F(x))$

$$\boxed{\alpha\times\beta=\sum_{x\in X_{(p)}}(i_x)_*(\alpha_x\cdot h_x^*(\beta))}$$

$\alpha\in C_p(X);\ \beta\in C_q(Y)$

$\alpha_x\in K_*(F(x))$

$v\mapsto x_0$, $v\mapsto y_0$, $p+q$

$$\Big(\sum_{x\in X_{(p)}}(i_x)_*(\alpha_x\cdot h_x^*\beta)\Big)_v=$$

$$=\sum_{\substack{x\in X_{(p)}\\ y'\in (Y_x)_{(q)}}}(i_{x*})^{y'}_v(\alpha_x\cdot h_x^*(\beta))=$$

$\begin{cases} i_x(y')=v \\ x=x_0\end{cases}$ $\to$ y' - общ. точка $(Y_{x_0})_{y_0}$, $h_{x_0}(y')=y_0$

$$=\sum_{\substack{y'-\text{о.т.}(Y_{x_0})_{y_0}\\ y'\mapsto v}}(i_{x_0})_*{}^{y'}_v\left(\alpha_{x_0}\,\ell(\mathcal{O}_{(Y_{x_0})_{y_0},y'})\cdot r_{F(y')/F(y_0)}(\beta_{y_0})\right)$$

$$(Y_{x_0})_{y_0}=Y_{x_0}\times_Y\operatorname{Spec}F(y_0)=$$
$$=\operatorname{Spec}(F(x_0)\otimes F(y_0))$$

2'°) $$\alpha\times\beta=\sum_{y\in Y_{(q)}}(j_y)_*(u_y^*(\alpha)\cdot\beta_y)$$

$$\begin{tikzcd} X_y=X\times\operatorname{Spec}F(y) \arrow[r, "j_y"] \arrow[d] \arrow[rr, bend left, "u_y"] & X\times Y \arrow[r] \arrow[d] & X \\ \operatorname{Spec}F(y) \arrow[r] & Y & \end{tikzcd}$$

3°) 50.2

$\alpha\in C_*X;\ \beta\in C_*Y;\ \gamma\in C_*Z$ $\quad (\alpha\times\beta)\times\gamma=\alpha\times(\beta\times\gamma)$ эл-тов $C_*(X\times Y\times Z)$

Лемма 102.1–102.2

B, C – комм лок. арт; ${}_BC$ – плоск.

а) ${}_BM$ – к. пор $\Rightarrow \ell_C(M\otimes_B C)=\ell(C/m_BC)\,\ell_B(M)$

б) $\ell(C)=\ell(C/m_BC)\cdot\ell(B)$

а) ⇒ б) $M:=B$ а) $\quad \ell_C(C/m_BC)=\ell(C/m_BC)$

$M=B/m_B$ $\quad M\otimes_B C=C/m_BC$

$\omega \in (X \times Y \times Z)_{p+q+r}$

$$\omega \mapsto v \mapsto x \in X_{(p)}, \quad v \mapsto y \in Y_{(q)}, \quad \omega \mapsto z \in Z_{(r)}$$

$$((\alpha\times\beta)\times\gamma)_\omega = \underbrace{\ell(\Theta_{Spec(F(v)\otimes F(z));p_\omega})}_{\ell_\omega}\, \tau_{F(\omega)/F(v)}((\alpha\times\beta)_v)\cdot \tau_{F(\omega)/F(z)}(\gamma_z) =$$

$$= \ell_\omega \underbrace{\ell(\Theta_{Spec(F(x)\otimes F(y));p_v})}_{\ell_v}\cdot \tau_{F(\omega)/F(x)}(\alpha_x)\,\tau_{F(\omega)/F(y)}(\beta_y)\,\tau_{F(\omega)/F(z)}(\gamma_z)$$

$C = F(X)\otimes F(y)\otimes F(z)_{p_\omega}$ $\quad s\otimes 1 \in p_\omega$

$\uparrow (-\otimes 1)$ $\quad \curvearrowright s \in p_v$

$B = F(x)\otimes F(y)_{p_v}$

$m_B = Bp_v \quad B/m_B = F(v)$

$C/m_B C = F(v)\otimes F(z)_{\overline{p_\omega}}$

$\ell_\omega \cdot \ell_v = \ell(C)$ $\square$

4° $\forall\, \alpha \in C_{p,n}(X);\ \beta\in C_{q,m}(Y)$

$$d_{X\times Y}(\alpha\times\beta) = d_X(\alpha)\times\beta + (-1)^{p+n}(\alpha\times d_Y(\beta))$$

$\alpha \in K_{p+n}(F(x));\ \beta\in K_{q+m}(F(y))$

$z \in (X\times Y)_{p+q-1}$

z. комп. всех 3 членов $=0$, если $\left[\begin{array}{l} pr_X(z)\notin \overline{\{x\}} \\ pr_Y(z)\notin \overline{\{y\}}\end{array}\right.$

$\left\{\begin{array}{l} pr_X(z)\in\overline{\{x\}} \\ pr_Y(z)\in\overline{\{y\}}\end{array}\right. \Rightarrow \left[\begin{array}{l} \boxed{pr_X(z) = x} \\ pr_Y(z) = y\end{array}\right.$ $\quad (d_X(\alpha)\times\beta)_z = 0$

$pr_X(z)\in X_{(p-1)}$

$$\begin{tikzcd} Y_x \arrow[r, "i_x"] \arrow[dr, "h_x"'] & X\times Y \arrow[d, "pr_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} Y_x \arrow[r] \arrow[rr, bend left, "i_x"] & \overline{\{x\}}\times Y \arrow[r, hook] & X\times Y \end{tikzcd}$$

локализация

а) $d_{X\times Y}(i_x)_* = (i_x)_* d_{Y_x}$

б) $(d_{X\times Y}(\alpha\times\beta))_z = d_{X\times Y}((i_x)_*(\alpha\cdot h_x^*\beta))_z \overset{49.5}{=}$

$$= (-1)^{p+n}[(i_x)_*(\alpha\cdot \underbrace{d_{Y_x}(h_x^*(\beta))}_{h_x^* d_Y(\beta)})]_z = (-1)^{p+n}(\alpha\times d_Y(\beta))_z$$

5° $X \xrightarrow{f} X'$; $Y \xrightarrow{g} Y'$ $(f\times g)_*(\alpha\times\beta) =$
$\alpha\in C_p(X)$; $\beta\in C_q(Y)$ $= f_*(\alpha)\times g_*(\beta)$

Д-во: $X = X'$ $f = id_X$ $x\in X_{(p)}$

$$\begin{tikzcd}
Y_x \arrow[rr, "i_x"] \arrow[dd, "g_x"'] \arrow[dr, "h_x"] & & X\times Y \arrow[dl, "pr_Y"'] \arrow[dd, "id\times g"] \\
& Y \arrow[dd, "g"] & \\
Y'_x \arrow[rr, "i'_x"] \arrow[dr, "h'_x"'] & & X\times Y' \arrow[dl, "pr_{Y'}"] \\
& Y' &
\end{tikzcd}$$

$$\begin{cases} (id\times g)\, i_x = i'_x\, g_x \\ g\, h_x = h'_x \cdot g_x \end{cases}$$

$$(id\times g)_*(\alpha\times\beta) = (id\times g)_*\Big(\sum_{x\in X_{(p)}} (i_x)_*(\alpha_x \cdot h_x^*(\beta))\Big) =$$

$$= \sum_x (i'_x)_* \circ (g_x)_*(\alpha_x\cdot h_x^*(\beta)) =$$

$\leftarrow K_*(F(x))$-mod. homo

$$= \sum_x (i'_x)_*(\alpha_x \cdot (g_x)_* h_x^*(\beta)) =$$

$$\overset{49.20}{=} \sum_x (i'_x)_*(\alpha_x \cdot h'^*_x(g_{x*}(\beta))) = \alpha\times g_*(\beta)$$

След. свойства:

6° $X' \xrightarrow{f} X$; $Y' \xrightarrow{g} Y$ – м $\forall \alpha\in C_p X$; $\beta\in C_q Y$

$$(f\times g)^*(\alpha\times\beta) = f^*(\alpha)\times g^*(\beta)$$

7° $\alpha\in C_*(X)$ $pr_X^*(\alpha) = \alpha\times[Y]$

$1\in\mathbb{Z}\subset K^M_*(F(Y))$

$$X\times Y \xrightarrow{pr_X} X$$

8° $\alpha\in C_p(U)$; $\beta\in C_q(Y)$

$$Z \hookrightarrow X \hookleftarrow U = X\setminus Z$$

$$\partial^U_Z(\alpha)\times\beta = \partial^{U\times Y}_{Z\times Y}(\alpha\times\beta)$$

9° $a\in k[X]^\times$ $\alpha\in C_p(X)$; $\beta\in C_q(Y)$

$$(\{a\}\alpha)\times\beta = \{a'\}(\alpha\times\beta)$$

$$a' = pr_X^*(a) = a\cdot pr_X \in F(X\times Y)^\times$$