

Finite-dimensional approximation

$$\left[\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right] \quad f \rightsquigarrow \left\{ f(x_i) \right\}_{x_i}$$

x_i

H - Cameron-Martin Hilbert space

H_n in some sense H_n is close to H
in C^k

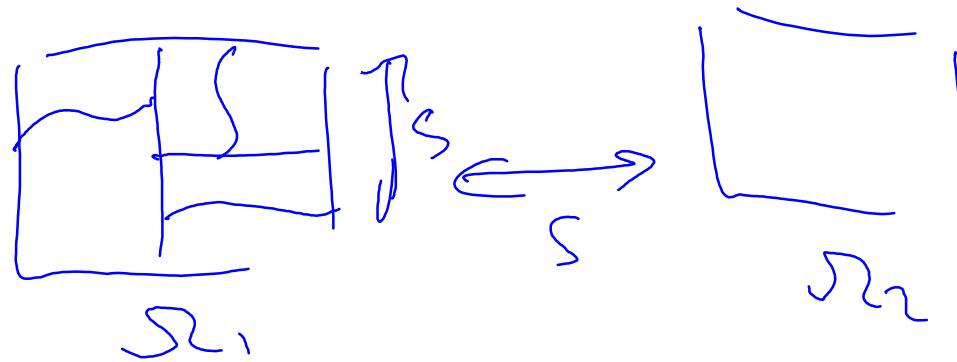
Ex $f = \sum a_i \varphi_i$ consider $V_n = \text{span} \{ \varphi_i \}_{i=1}^n$

$$f_n = \sum^n a_i \varphi_i \quad f_n \text{ is close to } f$$

$$K = \sum \varphi_i(x) \varphi_i(y)$$

Riverser - Venneville

in \mathbb{R}^2



$$A_i \in \sigma(S_i)$$

A_i events
generated
by crossings.

$$\gamma = \sup_{\substack{x \in S_1 \\ y \in S_2}} K(x-y)$$

$$|P(A_1 \cap A_2) - P(A_1)P(A_2)| \leq$$

$$\leq \frac{\gamma \Gamma(\text{Area } S_i + \text{length } \partial S_i)}{K(x)}$$

$$K(x) \sim |x|^{-\alpha} \quad \text{then} \quad S^{-\alpha} \quad S^{-4}$$

Quasi-ind. $\alpha > 4$

B - Muirhead - Rivera '18

PR⁵

X_t stationary process in \mathbb{R}

$$\alpha(s) = \sup_{\substack{A \in \mathcal{F}(-\infty, s) \\ B \in \mathcal{F}(s, \infty)}} |P(A \cap B) - P(A)P(B)|$$

Strong mixing : $\alpha(s) \rightarrow 0$ as $s \rightarrow \infty$

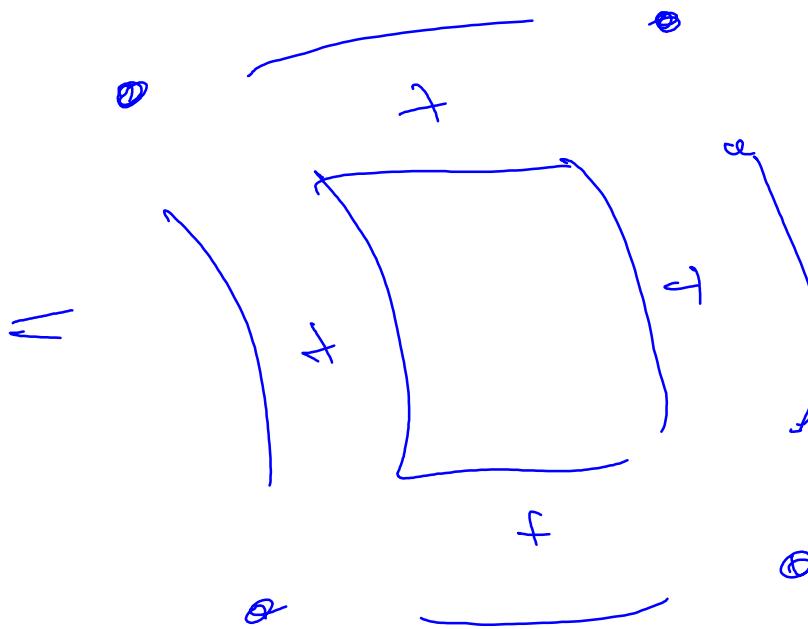
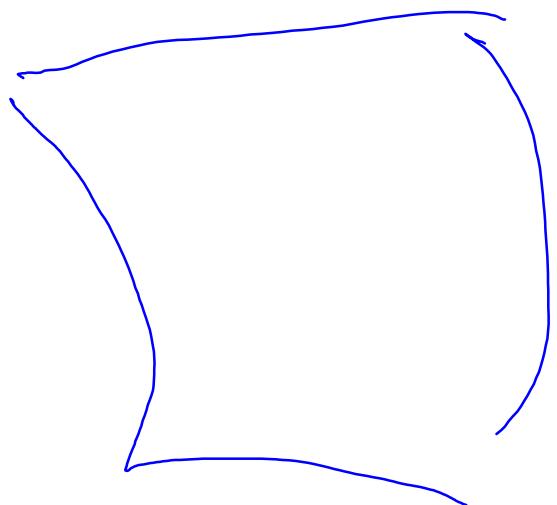
Kolmogorov-Rozanov '66 $\alpha \rightarrow 0$

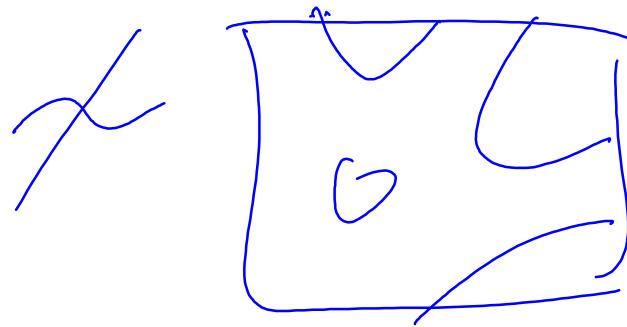
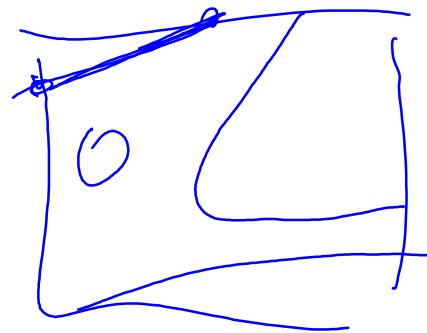
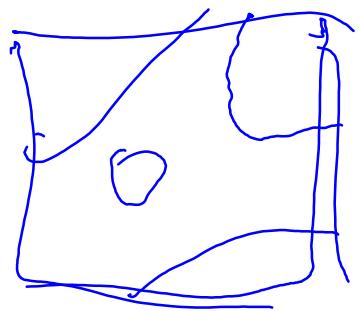
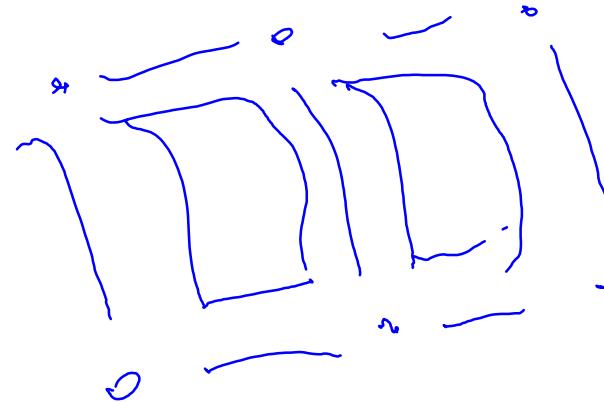
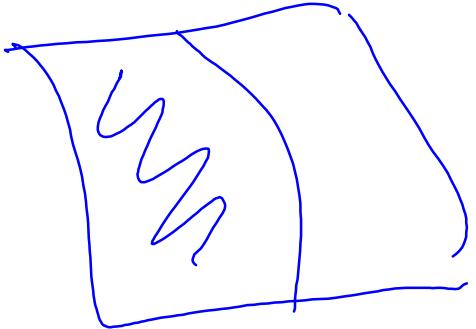
then $\int \frac{\log g(t)}{t+t^2} dt > -\infty$ g is spectral density

Note No strong mixing for analytic processes

~~gives complete knowledge of f on $(-\infty, 0)$~~

complete knowledge of f on $(-\infty, 0)$
gives complete knowledge of f on \mathbb{R}





The A_i are topological events
in stratified sets B_i

$$P(A \cap A_2) - P(A)P(A_2) =$$

$$= \int K(x, y) \underbrace{\left(d\pi^+(x, y) - d\pi^-(x, y) \right)}_{}$$

$B_1 \times B_2$ on each stratum

π^\pm are positive and cont w.r.t
Lebesgue measure on this stratum

densities are given by explicit
formulas in terms of K and its
derivatives

$$\underline{\text{Cor}} \quad T_{\text{top}}(B_1, B_2) \leq C \int_{B_1 \times B_2} K(x-y) dx dy$$

(f is stationary)

$$K \leq |x|^{-d}$$

$$\leq \text{Area } B_1 \times \text{Area } B_2 \times \\ \text{dist}(B_1, B_2)^{-d}$$

Cor If A_i are increasing

$$\text{then } \bar{\pi} = 0 \Rightarrow \text{FKG}$$

$$X \sim N(0, I_n), \bar{X} \text{ indep } N(0, I_n)$$

$$Z_t = (X, tX + \sqrt{1-t^2}\bar{X})$$

$$\frac{d}{dt} P[Z_t \in A \times B] = \int \langle \gamma_A(x), \gamma_B(y) \rangle dx dy$$

$\partial A \times \partial B$

$$P(Z_t \in A \times B) - P(Z_0 \in A \times B) = \int' \int' \langle \gamma_A(x), \gamma_B(y) \rangle dx dy$$

$\partial A \times \partial B$
 dt

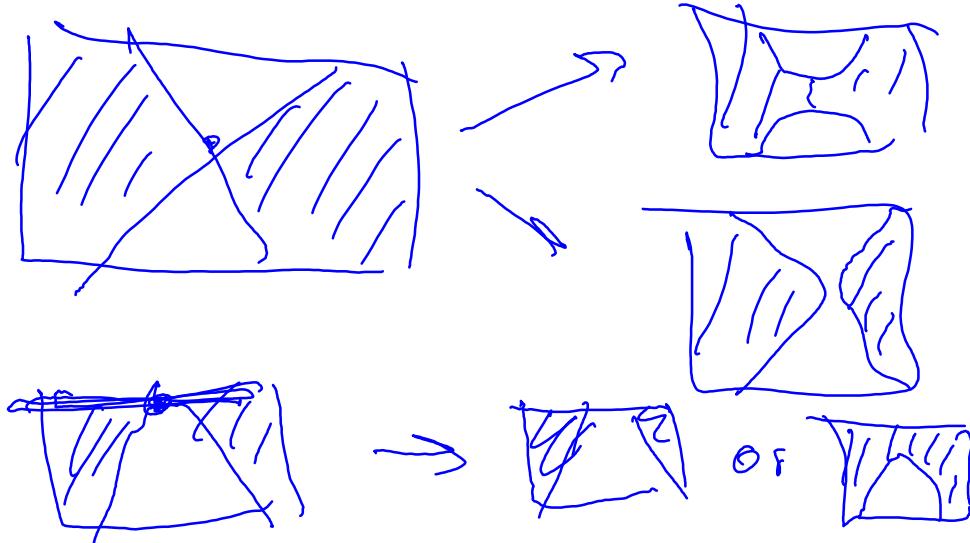
$$Z_1 = (X, X)$$

$$P(X \in A \cap B)$$

$$Z_0 = (X, \tilde{X})$$

$$P(X \in A) P(X \in B)$$

What is the boundary
of a top. event



Number of nodal domains

✓ ① E, LLN

? ② Var

③ CLT

Two complementary results

Th [Nazarov-Sodin'20] if stationary G, F
 $f \in C^3$ a.s., $K(x) = o(|x|^{-\alpha})$ for some $\underline{\alpha} > 0$

Then $\exists \sigma > 0$ s.t

$$\text{Var } N(B(R), f) \gtrsim R^\sigma$$

$$E \sim R^2 \quad \text{const} < \text{Var} \lesssim R^4$$

Conj For 'generic' field $\text{Var} \approx \mathbb{R}^2$

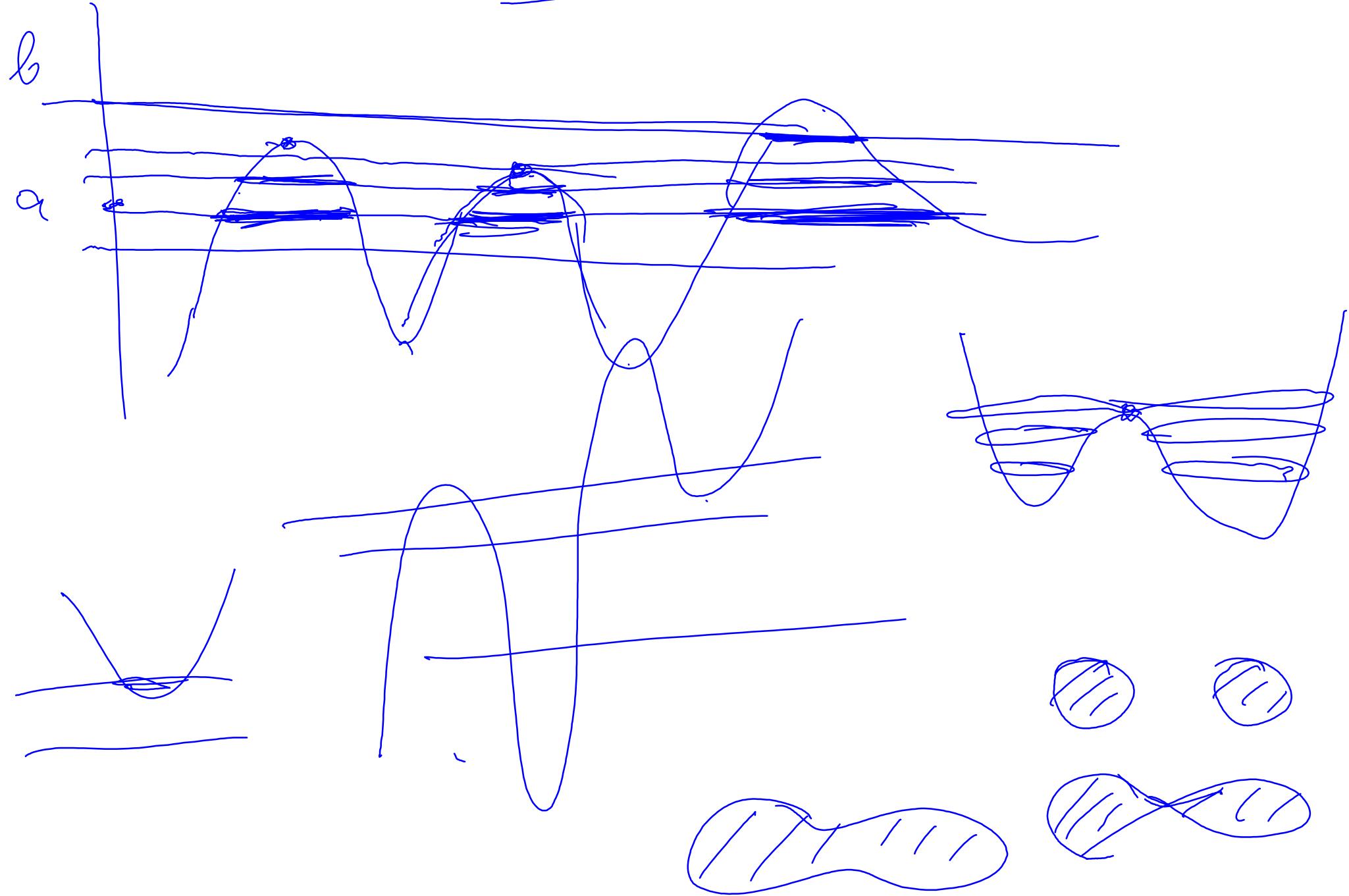
$\overbrace{\text{Th}\{B - \text{McAuley - Muirhead}'18', '19', '20\}}$

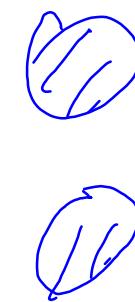
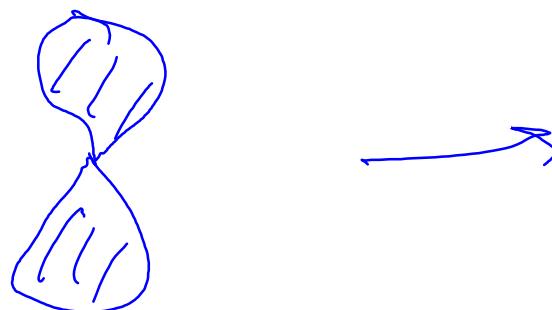
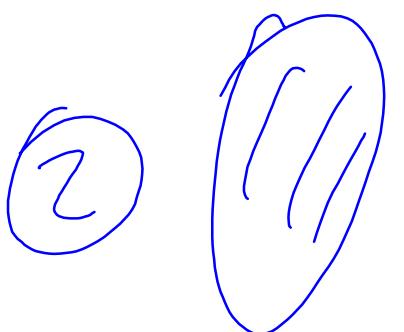
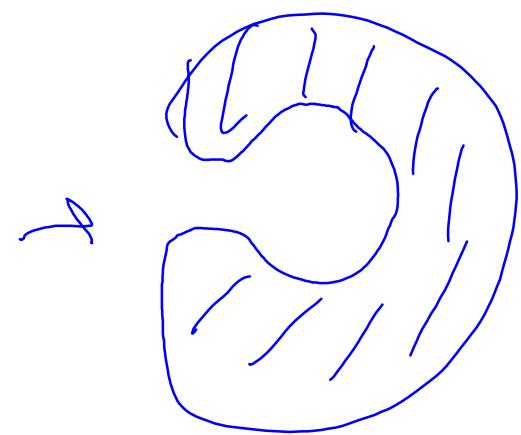
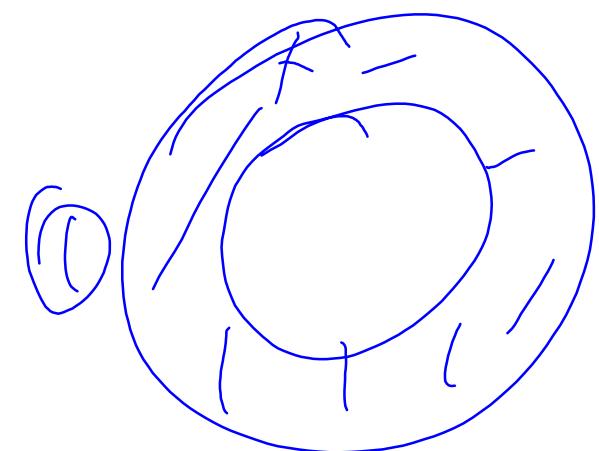
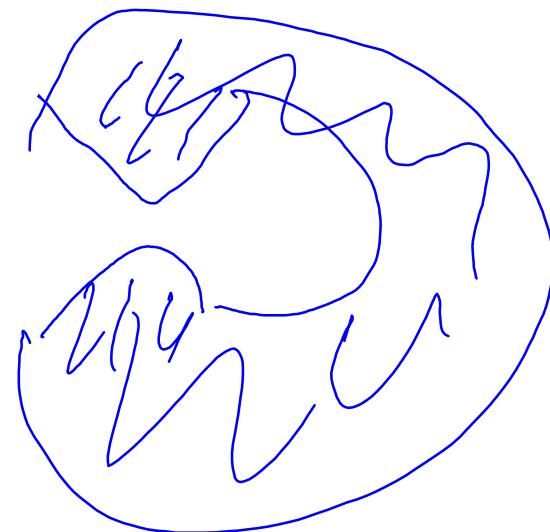
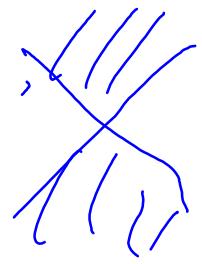
$$\frac{N_{ES}(B(R), l)}{\pi R^2} \rightarrow C_{ES}(l) \quad \frac{N_{LS}(B(R), l)}{\pi R^2} \rightarrow C_{LS}(l)$$

C_{ES}, C_{LS} can be written in terms of densities of critical points

$\underline{C_{ES}, C_{LS}}$ are cont. diff. functions of l

Morse theory





If $C'_{ES}(l) \neq 0$ then $\text{Var}(N_{\pm S}) \gtrsim R^2$

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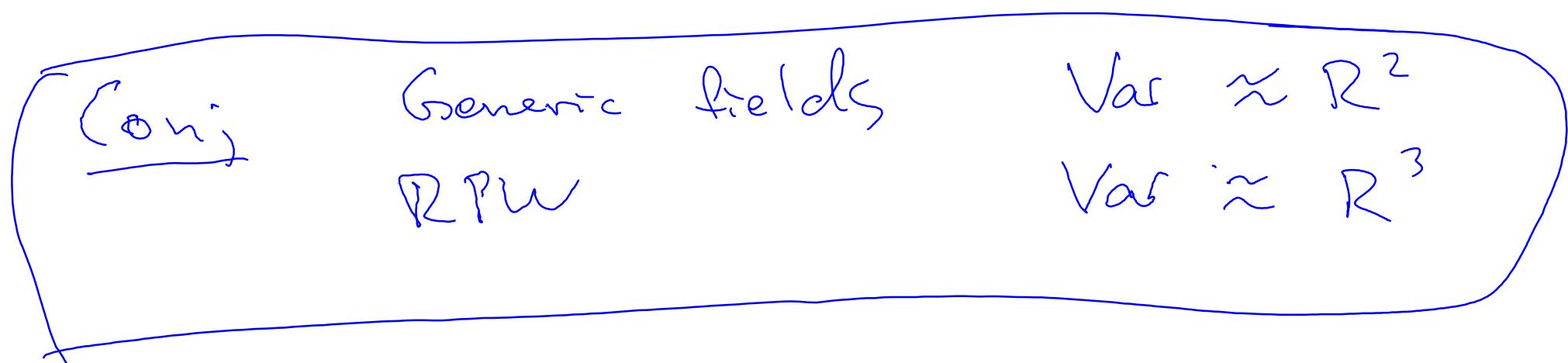
Symmetry $\Rightarrow C_{LS}(l) = C_{LS}(-l)$
 $\Rightarrow C'_{LS}(l) = 0$

Assumption If spectral measure has density near the origin and this density is unif. away from 0 in a nbhd of the origin

Not true for RPW

The B-M-M
For RPW if $\ell \neq 0$ and $\frac{C'_{ES}(\ell) \neq 0}{C'_{CS}(\ell) \neq 0}$

then $\text{Var } N_{ES} \gtrsim R^3$ | Note $C'_{ES}(0) \neq 0$
 $\text{Var } E_{CS} \gtrsim R^3$

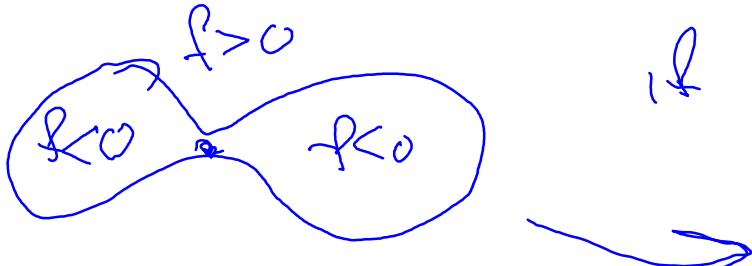


Nazarov-Sodin (main idea)

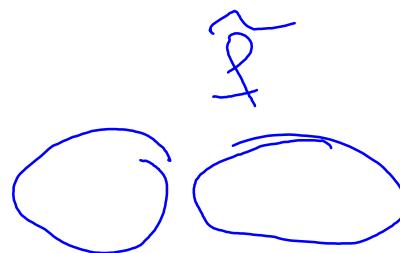
① There is a level s.t. there are many critical points in $B(R)$ with $x_i \in B$ s.t. $\nabla f = 0$ ($\#$ of all crit points $\asymp R^2$) s.t. $|f| < \varepsilon$ ($\asymp \varepsilon R^2$)

$\mathbb{E}(\#$ crit points $|f| < \varepsilon > R^\alpha$)
But they all are well separated (dis $> R^\beta$)

$$\tilde{f} = \sqrt{1 + \varepsilon^2} f + \varepsilon g$$

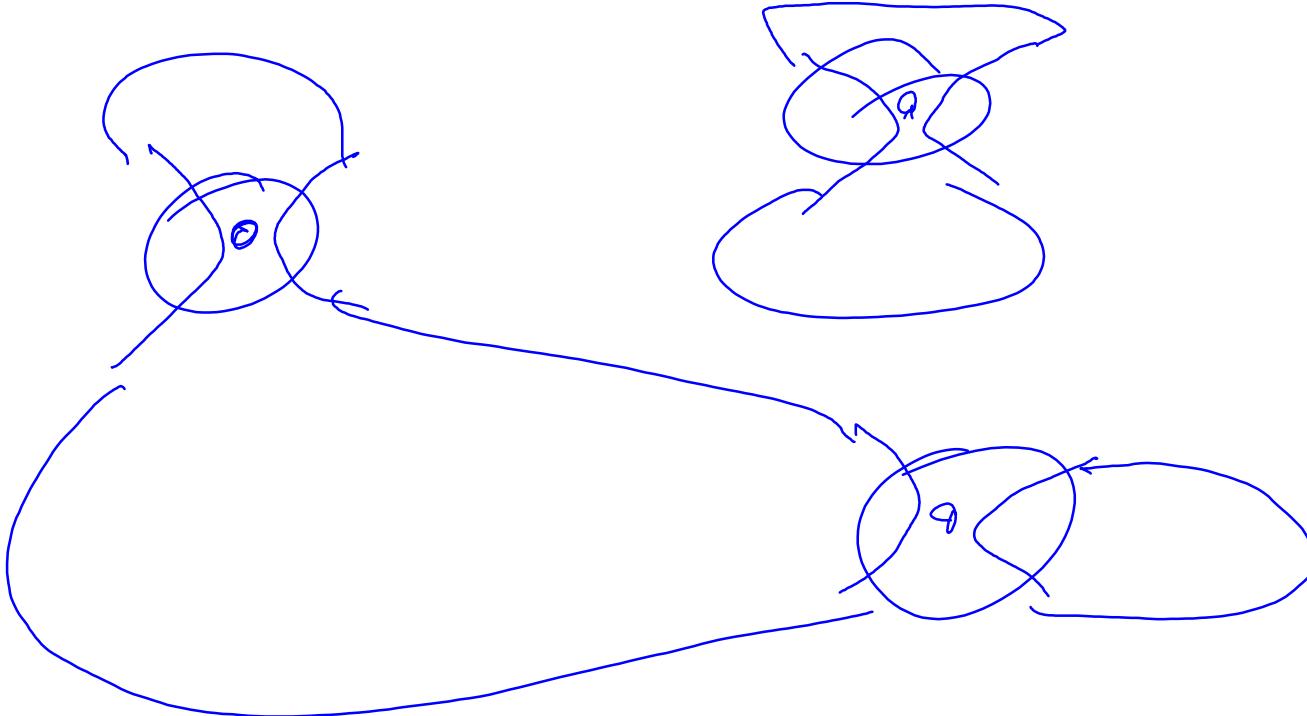


if $g > 0$ at low crit. point



g is an independent copy

(3)



almost independent percolation on
the collection of loops of size R^σ

\Rightarrow variance of the # loops $\geq R^\sigma$

B-M-4

Chatterjee Lemma X_n, Y_n r.v. $u_n \geq 0$

$$\exists c_1, c_2 \text{ s.t } \underline{\mathbb{P}(|X_n - Y_n| \geq c_1 u_n)} \geq c_2 > 0$$

$$d_{TV}(X_n, Y_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

then X_n has fluctuations of order at least u_n , in particular $\text{Var} X_n \gtrsim u_n^2$

$$X_R = N_{\bar{E}_S}(B(R), \ell)$$

$$Y_R = N_{\bar{Z}_S}(B(R), \ell + a_R)$$

$$c_{\bar{E}_S}(\ell) \neq 0 \Rightarrow X_R - Y_R \sim R^2 a_R \quad \Rightarrow u_n = R$$

$$d_{TV}(X_R, Y_R) \leq d_{TV}(\ell, \ell + a_R) \rightarrow 0$$

$$\text{if } a_R \ll \frac{1}{R}$$

$$\text{RPW} \quad f = \sum e^{\text{int}} a_n j_n(r)$$

in $B(R)$

$f \quad n > 2R$

$f \approx \left(\sum_{n=1}^{2R} \right) + \text{exponentially small.}$

$$d_{TV}(X_R, Y_R) \leq d_{TV}(f, \frac{f}{l+a_R} f) \approx$$

$$\approx d_{TV}\left(N(0, I_R), N\left(0, \left(\frac{l}{l+a_R}\right)^2 I_R\right)\right) \lesssim R \alpha_R^2$$

$$\alpha_R \sim o \frac{1}{\sqrt{R}} \Rightarrow \text{fluctuations} \sim \frac{R^2}{\sqrt{R}} \sim R^{3/2}$$

$$\Rightarrow \text{Var} \gtrsim R^3$$