

Def \mathbb{R}^n random variable $X = (X_1, \dots, X_n)$

Gaussian if $\forall y \in \mathbb{R}^n$ $\langle X, y \rangle$ is

Gaussian $\mu = (EX_1, \dots, EX_n)$

$C = (\text{Cov}(X_i, X_j))_{ij}$

$X \sim \mathcal{N}(\mu, C)$

Note $AX \sim \mathcal{N}(A\mu, A^T C A)$

Important Gaussian regression

$(X^1, X^2) \in \mathbb{R}^{n+m}$

$\mu = (\mu^1, \mu^2)$

$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

Then $X^1 | X^2 \sim$

$\mathcal{N}(\mu^1 + C_{12} C_{22}^{-1} (X^2 - \mu^2),$

$C_{11} - C_{12} C_{22}^{-1} C_{21})$

Def T is a set a Gaussian field is
a collection of Gaussian r.v $\{f_t = f(t)\}_{t \in T}$
 $\forall t_1, \dots, t_n, (f(t_1), \dots, f(t_n))$ is Gaussian
 $m(t) = \mathbb{E}f(t), \quad K(s, t) = \mathbb{E}[(f(t) - m(t))(f(s) - m(s))]$

Consider $\varphi_1, \dots, \varphi_n : T \rightarrow \mathbb{R}$

$$f(t) = \sum a_i \varphi_i(t) \quad a_i \text{ iid } N(0, 1)$$

f is a Gaussian field

$$K(s, t) = \sum \varphi_i(s) \varphi_i(t)$$

$H = \text{span}\{\varphi_i\}$ with $\langle \cdot, \cdot \rangle_H$ s.t.
 $\{\varphi_i\}$ o.n.b

$f \sim (a_1, \dots, a_n) \leftarrow$ standard Gaussian vector in \mathbb{R}^n

f is the standard G.V. in H

Let K be a covariance kernel of f

Goal: Construct H

$K_s(t) = K(s, t)$, consider $\text{span}\{K_s\}$ set

$$\left\langle \sum a_i K_{t_i}, \sum b_j K_{s_j} \right\rangle \stackrel{\text{def}}{=} \sum K(t_i, s_j) a_i b_j$$

H is the completion of $\text{span} \{k_s\}$
w.r.t to $\langle \cdot, \cdot \rangle_H$

f is cont, K cont.

$$\text{span} \{k_s\} \subset C(T)$$

$$H \hookrightarrow C(T)$$

Let $h \in H$, define $h(t) \stackrel{\text{def}}{=} \langle h, k_t \rangle_H$

$$\|k_t - k_s\|^2 = K(t,t) + K(s,s) - 2K(t,s) \rightarrow 0$$

$$|h(t) - h(s)| \leq \|h\| \|k_t - k_s\| \rightarrow 0$$

H is the Cameron-Martin space

Let $\{\varphi_i\}$ o.n.b in H

$h = \sum \langle h, \varphi_i \rangle \varphi_i \leftarrow$ conv in H

$$\left| h(t) - \sum_{i=1}^N \langle h, \varphi_i \rangle \varphi_i(t) \right| = \left| \langle h - \sum_{i=1}^N \langle h, \varphi_i \rangle \varphi_i, \varphi_N \rangle \right|$$

$$\leq \underbrace{\| h - \sum_{i=1}^N \langle h, \varphi_i \rangle \varphi_i \|}_{\sqrt{K(t,t)}} \underbrace{\| \varphi_N \|}_{\sqrt{K(t,t)}} \rightarrow 0 \text{ as } N \rightarrow \infty$$

$\sum \langle h, \varphi_i \rangle \varphi_i(t) \rightarrow h(t)$ locally uniformly

Apply to $h = k_s$

$$k_s(t) = k(s,t) = \sum \langle k_s, \varphi_i \rangle \varphi_i(t) = \sum \varphi_i(s) \varphi_i(t)$$

con. loc. unit.

\mathcal{F} Gaussian field $\mathcal{H} = \text{span} \{ f(t), t \in T \}$
in $L^2(\mathbb{P})$

Define: $\Phi(\sum a_i k_{t_i}) = \sum a_i f(t_i)$
 \uparrow Gaussian r.v.

$$\begin{aligned} \|\Phi(\sum a_i k_{t_i})\|_{\mathcal{H}}^2 &= \mathbb{E} \left(\sum a_i f(t_i) \sum a_j f(t_j) \right) \\ &= \sum a_i a_j k(t_i, t_j) = \|\sum a_i k_{t_i}\|_{\mathcal{H}}^2 \end{aligned}$$

Φ is norm preserving \Rightarrow extend to closures

If $\{\varphi_i\}$ o.u.b in H
 Define $\zeta_i = \Phi(\varphi_i) \in \mathcal{H}$ $\{\zeta_i\}$ o.u.b in \mathcal{H}
 $\Rightarrow \zeta_i$ are iid $N(0,1)$

$$f = \sum \zeta_i \langle \zeta_i, f(t) \rangle_{L^2} = \sum \zeta_i \underbrace{E[\zeta_i f(t)]}$$

\uparrow
 conv in $L^2(\mathbb{P})$

$$\langle \varphi_i, K_t \rangle_H = \varphi_i(t)$$

$$= \underbrace{\sum \zeta_i \varphi_i(t)}$$

\nwarrow diverges in H

$$\| \sum \zeta_i \varphi_i(t) \|_H^2$$

$$= \sum \|\zeta_i\|^2 \leftarrow \text{div. with probability 1.}$$

The IP f is a.s. cont. then
 $f(t) = \sum \gamma_i \Psi_i(t)$ conv. loc. unif. with prob. 1

Example $T = \{1, \dots, n\}$ $X = (X_1, \dots, X_n)$
 cov $C = (C_{ij})$ $C(i, i) = C_{ii}$

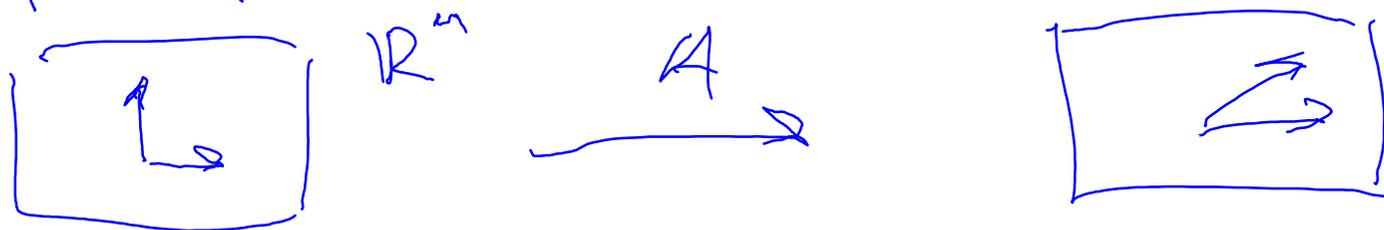
Scalar product in RKHS

$$\langle f, g \rangle_H := \sum f_i g_j C_{ij}^{-1}$$

$$\langle f, C(k, \cdot) \rangle_H = \sum_{ij} f_i C_{kj} C_{ij}^{-1} = \sum_i f_i \delta_{ki} = f_k$$

$$\langle f, g \rangle_H = \langle A^{-1} f, A^{-1} g \rangle_{\mathbb{R}^n} \quad A = C^{1/2}$$

$X \sim AW$ W is standard G.V. in \mathbb{R}^n



Different terminology

Def Let H be a real sep. Hilbert space
 $W: H \rightarrow L^2(P)$ is H -isnormal process
if it is a centered G.P. s.t

$$E W(h_1) W(h_2) = \langle h_1, h_2 \rangle_H$$

W is the white noise in H

Formally $W = \sum a_i \varphi_i$ $\{\varphi_i\}$ o.n.b a_i iid $N(0,1)$

So Gaussian field \sim isnormal process.

Ex $H = H^1(\Omega)$ Sobolev space

$$\langle h_1, h_2 \rangle = \int \langle \nabla h_1, \nabla h_2 \rangle dx$$

isnormal process \int_{Ω} is Gaussian Free Field
 $K(x,y) = G(x,y)$ Green's function in Ω

Ex $T = [0, 1]$ X_t standard Brownian motion.

$$H = \left\{ f : f(t) = \int_0^t f'(u) du, \int_0^1 (f')^2 < \infty \right\}$$

$$\langle f, g \rangle = \int_0^1 f' g'$$

Exercise Prove that this is RKHS.

Q Given V , is there f and what are its properties.

Kolmogorov's theorem \Rightarrow f exists

Th (Kolmogorov) $K: [0,1]^n \times [0,1]^n \rightarrow \mathbb{R}$

K is a positive def. function.

$\exists \alpha > 0, C_0$ s.t. $K(x,x) + K(y,y) - 2K(x,y) \leq \underline{C_0} |x-y|^{2\alpha}$

$\forall x, y \in [0,1]^n$ then $\exists!$ Gaussian measure

μ on $C([0,1]^n, \mathbb{R})$ s.t.

$\int f(x) f(y) \mu(d\omega) = K(x,y)$ ($\Leftrightarrow \mathbb{E} f(x) f(y) = K(x,y)$)

Moreover $\forall 0 < \beta < \alpha$ then

$\mu(C^\beta) = 1$

$f \in C^\alpha$

with probability 1.

"Proof"

① $B = C(\Sigma_{0,1}^n, \mathbb{R})$ B^*
point evaluations are dense in B^*

Need Gaussian measure μ s.t.
 $\forall x_i$ $(f(x_i))_i$ is Gaussian and

$$\int f(x) f(y) d\mu = K(x, y)$$

② $\exists \mu_0$ like this on $X = \mathbb{R}^{\Sigma_{0,1}^n}$

want: $\|X\|_\beta < \infty$ a.s

$\|\cdot\|_\beta$ depends on uncountably many valuations

$\Rightarrow \|\cdot\|_\beta$ not measurable

③ \mathcal{D} set of all dyadic points in $[0,1]^n$

$$\Omega_\beta = \left\{ f \in \mathcal{X} \mid \hat{f}(x) := \lim_{\substack{y \rightarrow x \\ y \in \mathcal{D}}} f(y) \text{ exists. } \forall x, \hat{f} \in C^\beta \right\}$$

depends on countably many evaluations \Rightarrow meas.

$$\Phi: \mathcal{X} \rightarrow C^\beta \quad \Phi(f) = \begin{cases} \hat{f} & f \in \Omega_\beta \\ 0 & f \notin \Omega_\beta \end{cases}$$

Need $\mu_0(\Omega_\beta) = 1 \quad \forall 0 < \beta < \alpha$

$$M_\beta(f) \stackrel{\text{def}}{=} \sup_{\substack{x \neq y \\ x, y \in \mathcal{D}}} \frac{|f(x) - f(y)|}{|x - y|^\beta} = \|f|_{\mathcal{D}}\|_\beta$$

Note $\Omega_\beta = \{ f: M_\beta(f) < \infty \}$

$$\mu_0(\Omega_\beta) = 1 \quad \text{if} \quad \mathbb{E} M_\beta(f) < \infty$$

$D_m \subset D$ points with const. $\frac{K}{2^m}$

$$\Delta_m = \left\{ (x, y), x, y \in D_m, |x - y| = \frac{1}{2^m} \right\}$$

$$|\Delta_m| \lesssim \frac{m n}{2^m}$$

Take $0 < \beta < \alpha' < \alpha$

$$K_m(f) = \sup_{x, y \in \Delta_m} |f(x) - f(y)|$$

$$\mathbb{E} K_m^p(f) \leq \sum_{x, y \in \Delta_m} \mathbb{E} |f(x) - f(y)|^p \lesssim \sum \left(\mathbb{E} |f(x) - f(y)|^2 \right)^{p/2}$$

$$= \sum (K(x, x) + K(y, y) - 2K(x, y))^{p/2}$$

$$\lesssim 2^{nm - \alpha m p} \lesssim 2^{nm - \alpha' m p}$$

$$\lesssim 2^{-\beta' m p}$$

assume that p is large enough

$$\boxed{\mathbb{E} K_m(f) \leq \text{const } 2^{-\beta' m}}$$

↑ By Jensen's inequality

Take $x \neq y \quad \exists m_0 \text{ s.t.}$

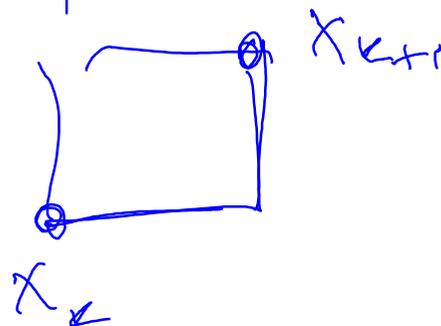
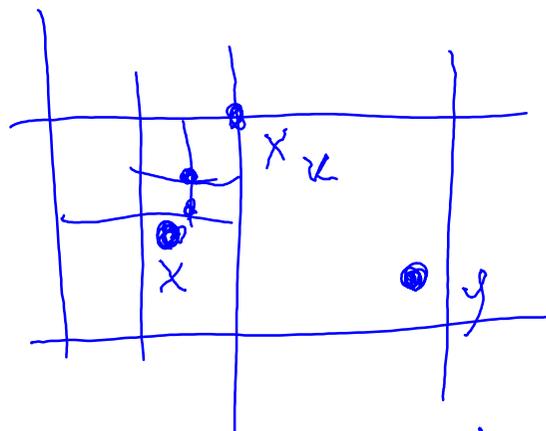
$$2^{-m_0-1} < |x-y| < 2^{-m_0}$$

$$\exists x_n \rightarrow x \quad x_n, y_n \in D$$

$$y_n \rightarrow y$$

If $k > m_0$ then

x_k, x_{k+1} are vertices
of the same dyadic cube
of size 2^{-k-1}



Then $|f(x) - f(y)| \leq |f(x_{m_0}) - f(y_{m_0})| +$

$$+ \sum_{m_0}^{\infty} (|f(x_k) - f(x_{k+1})| + |f(y_k) - f(y_{k+1})|)$$

$$\leq 2^n \sum_k K_k(f)$$

↑ two vertices of the same cube are
connected through at most n edges

$$M_{\beta}(f) \leq 2\eta \sup_{m \geq 0} 2^{\beta(m+1)} \sum_m k_m(f) \leq$$

$$\leq \sum 2^{\beta k} k_k(f)$$

$$\|M_{\beta}(f)\|_2 \leq \sum 2^{\beta k} \|k_k\|_2 \leq \sum_{k=0}^{\infty} 2^{\beta k - \beta' k} < \infty$$

