

$$f = \mathcal{F} W_g$$

$$dg = g dx$$

$$W_g = \int g W_{L^2}(dx)$$

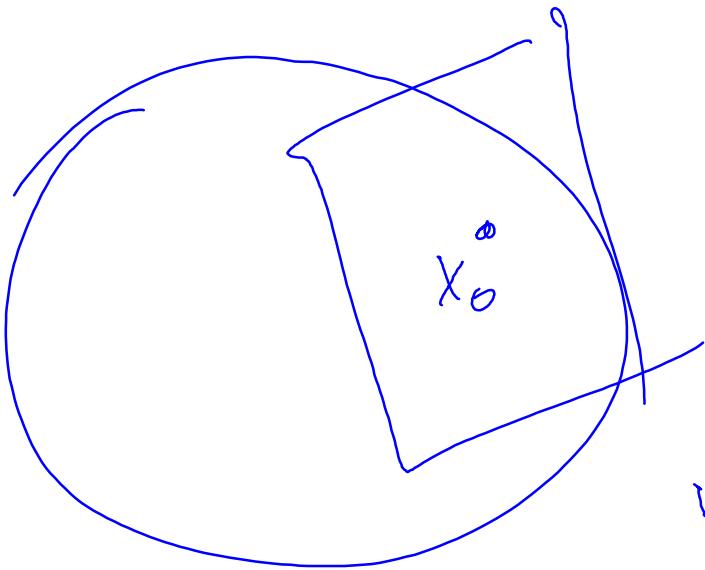
$$f = \mathcal{F} \int g W_{L^2} = (\mathcal{F} g) * \mathcal{F} W_{L^2} = \underbrace{g * W_{L^2}(dx)}$$

$$g * g = K - \text{cov. kernel}$$

$2n+1$ dimensional space of s.h. of deg n

equip with L^2

Pass to the limit as $n \rightarrow \infty$



$$F_n(u) := I_n \left(\exp_{x_0} \left(\frac{u}{\sqrt{n}} \right) \right)$$

Cov. Kernel of RSM

$$\text{is } P_n(\cos d_S(x, y))$$

$$P_n(\cos \theta) \sim \sqrt{\frac{\theta}{\sin \theta}} J_0 \left((n + \frac{1}{2}) \theta \right) + O\left(\frac{1}{n}\right)$$

Hilf's formula

$$\text{Cov of } F_n \quad K_n = P_n \left(\cos d_S \left(\exp_{x_0} \left(\frac{u}{\sqrt{n}} \right), \exp \left(\frac{v}{\sqrt{n}} \right) \right) \right)$$

ss $\frac{|u-v|}{\sqrt{n}}$

$$K_n(u, v) \approx J_0(|u-v|)$$

RSM \longrightarrow RPW

Homogeneous poly of deg n
Fubini-Study metric

$$\sqrt{n} \mathbf{x}^J \quad J = (j_1, j_2, j_3) \quad |J| = n$$

$$f = \sum a_J \sqrt{n} \mathbf{x}^J \quad a_J \sim N(0, 1)$$

$$K(x, y) = \cos(d(x, y))$$

$$F_n = f_n \left(\exp_{x_0} \left(\frac{x}{\sqrt{n}} \right) \right)$$

then

$$(x - y)^2 / 2$$

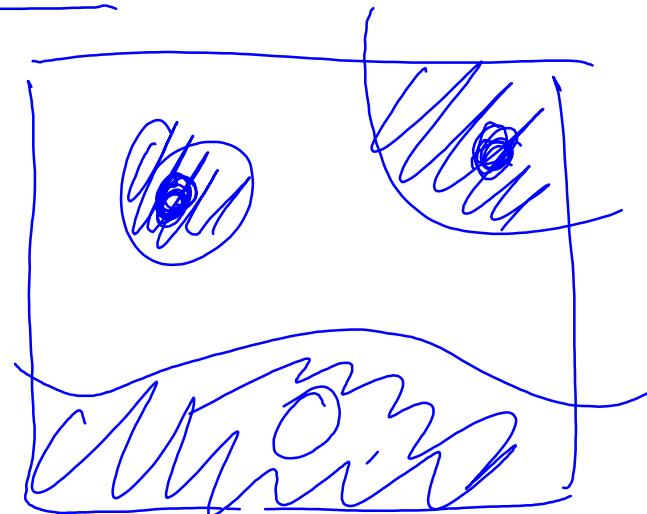
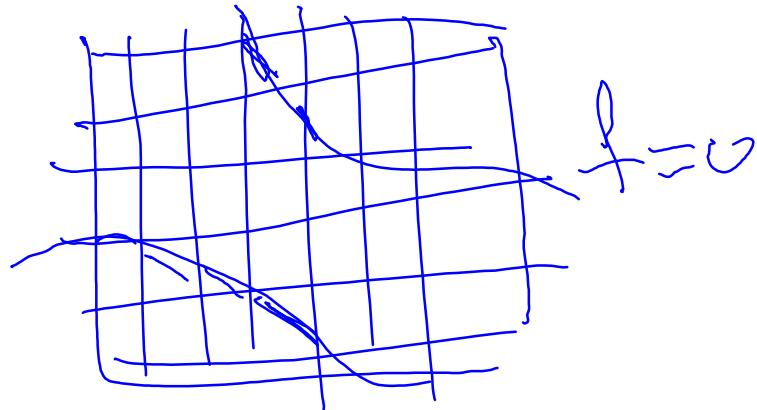
$$\text{cov of } F_n \xrightarrow[n \rightarrow \infty]{} e$$

$$e$$

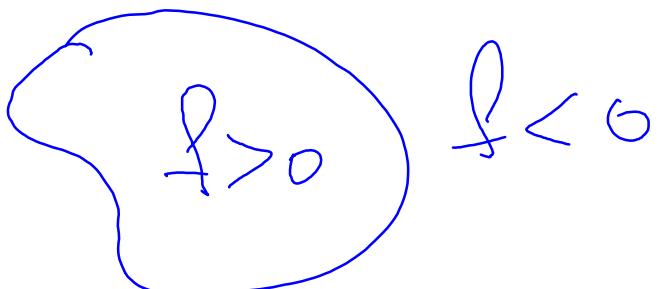
Bargmann-
Fock

Kac - Rice gives all local information

Non-local observables



Note



nodal domains \leq # critical points

There is a loc. extremum in every domain

What is the number of nodal domains of a RSM

Courant: $\sqrt{S.M.} \quad \# \text{ nodal domains} \leq n^2$

No lower bound

Lewy: $\exists \text{ S.M. with 1 or 2 nodal lines}$ (depending on parity of n)

The [Nazarov-Sodin '07] There is $a > 0$

$$\text{s.t. } P\left[\left|\frac{N(f_n)}{n^2} - a\right| > \varepsilon\right] \leq e^{-cn}$$

(dim of sphere is 2)

Th [Nazarov - Sodin'16] Let f be a stationary field with spectral measure \mathfrak{F} s.f.

① $\int |f|^{\frac{n}{n-1}+\epsilon} d\mathfrak{F} < \infty$ for some $\epsilon > 0$

② \mathfrak{F} has no atoms linear

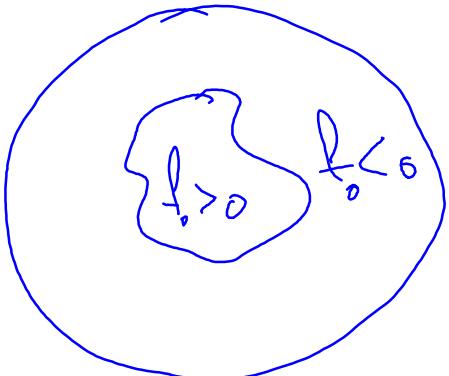
③ \mathfrak{F} is not supported on a hyperspace

Then $\exists a > 0 \forall$ open convex $S, \partial S$

$$\frac{N(f, R \cdot S)}{R^n \text{Vol}(S)} \xrightarrow{\text{a.s}} a \quad \text{and in } L^1$$

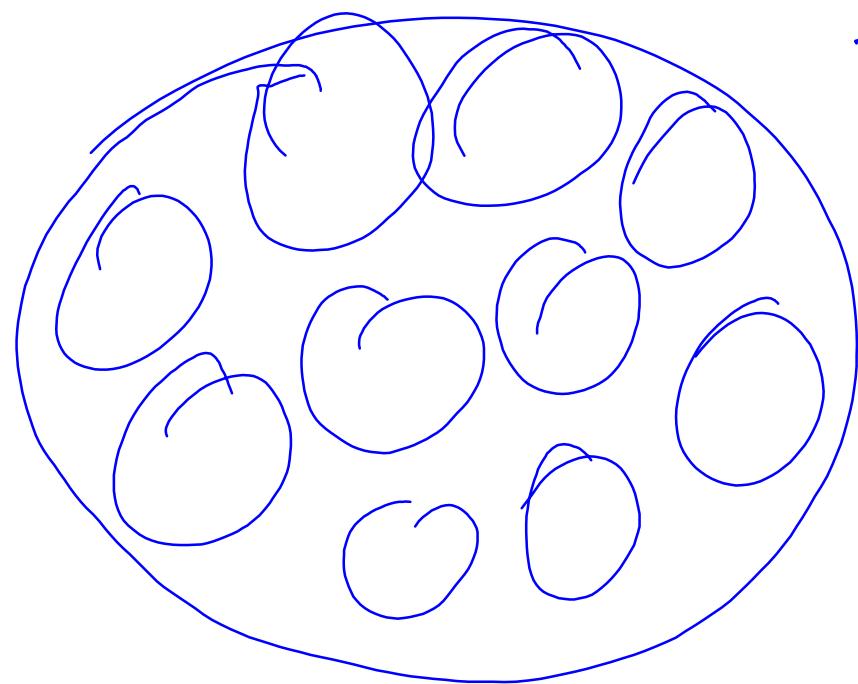
④ if $\text{supp } \mathfrak{F}$ contains an open set or a circle (centered at 0) then $a > 0$

- ① $\Rightarrow f$ is in C^2 a.s.
 ② Field is ergodic
 ③ Law of ∇f is non degenerate
 ④ $\exists R_0$ and $f_0 \in H$ s.t.
 f_0 has a nodal domain inside $B(0, R_0)$
 and $\nexists x \in B(0, R_0)$ s.t. $f(x) = 0, \nabla f(x) = 0$
 \Rightarrow Basis in H f_0, f_1, \dots



$f = \sum a_k f_k$ $\left\| \sum_{k=N}^{\infty} a_k f_k \right\|_{C^1(B)} \leq \varepsilon$
 $\exists N$ s.t. with prob almost 1
 with pos. prob c_0 is very large
 and a_1, \dots, a_{N-1} are very small

then $f = f_0 + \text{error} \Rightarrow$
 \Rightarrow nodal line of f is close to
 nodal line of f_0
 \Rightarrow With positive probability f has a
 nodal domain inside $B(R_0)$



$B(R)$ if inside
 $\sim \left(\frac{R}{R_0}\right)^n$ discs of
 radius R_0
 $\mathbb{E} \# \text{Nodal domains}$
 $\gg \text{const. } R^n$

$$\int_{B(R+r)} \frac{N(u, r, f)}{\text{Vol } B(r)} du \leq N(R, f) \leq \int_{B(R+r)} \frac{N^*(u, r, f)}{\text{Vol } B(r)} du$$

$N^*(u, r, f) = \# \text{ of nodal domains that intersect } B(u, r)$

$$\dots \leq \frac{N(R, f)}{\text{Vol } B(R)} \leq \left(1 + \frac{r}{R}\right)^n \frac{1}{\text{Vol } B(R+r)} \frac{\int_{B(R)} N^*}{\text{Vol } B(R)}$$

$$0 \leq N^* - N \leq \# \text{ of critical points}$$

$f|_S = \# \text{ zeros of } f|_S \approx \text{const. length}(S)$

- - - - -

$\frac{N(0, r, \tau_u f)}{\text{Vol } B(r)}$

$\xrightarrow{R \rightarrow \infty}$

$\int \frac{N(u, r, f)}{\text{Vol } B(r)} du$

$\boxed{\tau_u f = f(x+u)}$

$\boxed{\frac{\mathbb{E} N(0, r, f)}{\text{Vol } B(r)}}$

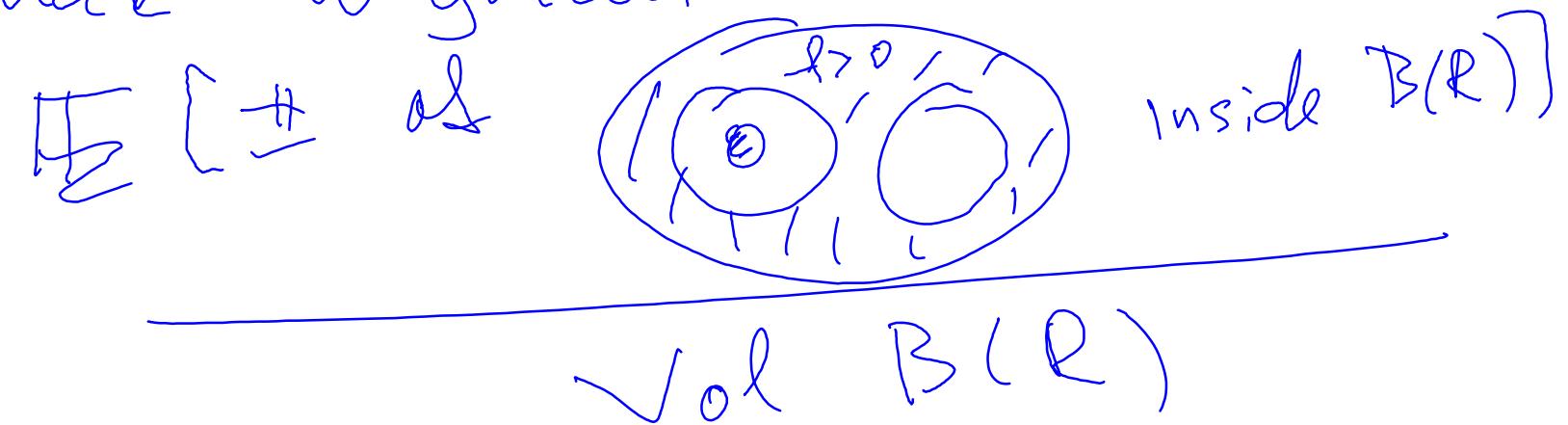
$\frac{\mathbb{E} N(0, r, f)}{\text{Vol } B(r)}$

$\lim_{R \rightarrow \infty} \frac{N(R, f)}{\text{Vol } B(R)} \leq \lim_{R \rightarrow \infty} \frac{V(R, f)}{\text{Vol } B(R)} \leq \frac{\mathbb{E} N(0, r, f)}{\text{Vol } B(r)}$

Sarnak - Wigman

$\exists \lim$

$R \rightarrow \infty$



B - Wigman

$E \{ \# \text{ of domains with}$

Area $\approx ?$,

perimeter $\gtrsim ?$

inside $B(r)$

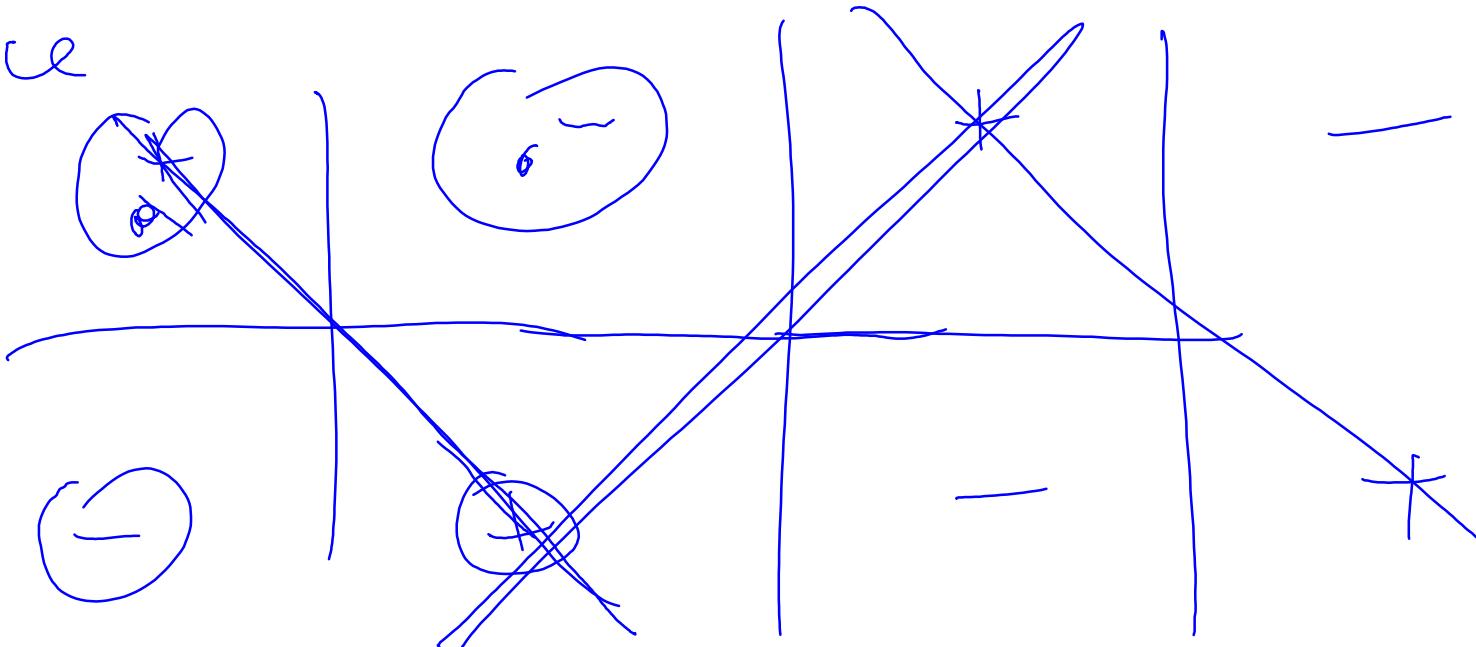
$\text{Vol } B(R)$

Q What is a?

Bogomolny-Schmit conjecture 2006

↪ typical spacing between nodal lines of RPW

nodal lines on average form a square lattice



Nodal domains of RPW
look like clusters in the
critical bond perc. on \mathbb{Z}^2
 S is given by the wavelength.

Physics \Rightarrow # clusters per vertex
is known

\rightarrow # of domains per unit volume