Random sections of convex bodies. Distribution of the volume of weighted Gaussian simplex

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Plan

1) Distance between two random points and length of the random chord (Euclidean case).

2) Distance between two random points and length of the random chord (Spherical case).

3) Distribution of the volume of weighted Gaussian simplex.

Random sections of convex bodies

Let K be a convex body in $\mathbb{R}^d$.

- $\Delta = \Delta(K)$ — distance between two random points independently and uniformly chosen in K
- $\sigma = \sigma(K)$ — length of the intersection of K with random line (length of the random chord of K)

Our goal — to find connection between $\Delta(K)$ and $\sigma(K)$.

Known results

Theorem (J.F.C. Kingman, 1969)

$$\int_{P_1,P_2\in K} \operatorname{dist}(P_1,P_2)^p \mathrm{d}P_1 \mathrm{d}P_2 =$$
$$= \frac{d\kappa_d}{(d+p)(d+p+1)} \int_{g\cap K\neq\emptyset} |g\cap K|^{p+d+1} \mu_{d,1}(\mathrm{d}g).$$

Connection between moments:

$$\mathbb{E}\,\Delta^p = \frac{\kappa_{d-1}}{(d+p)(d+p+1)} \cdot \frac{|\partial K|}{|K|^2} \mathbb{E}\,\sigma^{d+p+1}.$$

Known results

- $d = 2$: explicit formula for the CDF of $\Delta(K)$ via the CDF of $\sigma(K)$ — *Aharonyan N. G.* The Distribution of the Distance between Two Random Points in a Convex Set.

Main result

$$F_\Delta(t) = \mathbb{P}\{\mathrm{dist}(P_0, P_1) < t\}.$$

$$F_\Sigma(t) = \mathbb{P}\{|g \cap K| < t\}.$$

Theorem (T.M., 2019)

$$F_\Delta(t) =$$

$$= \frac{1}{|K|}\left[\frac{\omega_d t^d}{d} - \frac{|\partial K|}{|K|}\frac{\kappa_{d-1} t^{d+1}}{d+1} + \frac{|\partial K|}{|K|}\frac{\kappa_{d-1}}{d}\int\limits_0^t \left(t^d - s^d\right) F_\sigma(s)\mathrm{d}s\right].$$

Corollary

Corollary (T.M., 2019)

$$f_{\Delta}(t) = \frac{t^{d-1}}{|K|}\left(\omega_d - \frac{|\partial K|}{|K|}\kappa_{d-1}\int\limits_0^t (1 - F_{\sigma}(s))\,\mathrm{d}s\right).$$

Corollary (Kimgman formula)

$$\int_{P_1,P_2\in K} \mathrm{dist}(P_1,P_2)^p \mathrm{d}P_1\mathrm{d}P_2 =$$

$$= \frac{d\kappa_d}{(d+p)(d+p+1)}\int_{g\cap K\neq\emptyset} |g\cap K|^{p+d+1}\mu_{d,1}(\mathrm{d}g).$$

Corollary

$d = 2$. (Aharonyan N.G., 2015)

Blaschke-Petkantschin formula

h — non-negative measurable function, $k \leqslant d$

Theorem

$$\int\limits_{(\mathbb{R}^d)^{k+1}} h(x_0, \ldots, x_k) \mathrm{d}x_0 \ldots \mathrm{d}x_k =$$

$$= C_{d,k} \int\limits_{A_{d,k}} \int\limits_{E^{l+k}} h(x_0, \ldots, x_k) |\mathrm{conv}(x_0, \ldots, x_k)|^{d-k}$$

$$\times \lambda_E(\mathrm{d}x_0) \ldots \lambda_E(\mathrm{d}x_k) \mu_{d,k}(\mathrm{d}E).$$

Spherical Blaschke-Petkantschin formula

Theorem

$$\int\limits_{(\mathbb{S}^{d-1})^k} h(x_1, \dots, x_k)\lambda(\mathrm{d}x_1)\dots\lambda(\mathrm{d}x_k)$$

$$= C_{d,k} \int\limits_{G_{d,k}} \int\limits_{(E\cap\mathbb{S}^{d-1})^k} h(x_1, \dots, x_k)|\mathrm{conv}(0, x_1, \dots, x_k)|^{d-k}$$

$$\times \lambda_E(\mathrm{d}x_1)\dots\lambda_E(\mathrm{d}x_k)\mu_{d,k}(\mathrm{d}E)$$

Spherical case

$K \subset \mathbb{S}^{d-1}$ — spherical convex body
($K = \mathbb{S}^{d-1} \cap C$, C — line-free closed convex cone in $\mathbb{R}^d$.)

- $\Delta = \Delta(K)$ the spherical distance between two random points independently and uniformly chosen in K
- $\sigma = \sigma(K)$ to be the spherical length of the intersection of K with the random 2-plane.

Main result

Theorem (A.Tarasov, D.Zaporozhets, T.M., 2020)

For any spherical convex body $K \subset \mathbb{S}^{d-1}$, the density function of distribution of $\Delta(K)$ can be expressed in terms of the distribution function of $\sigma(K)$ as follows:

$$f_\Delta(t) = \frac{\sin^{d-2} t}{|K|}\left(\omega_{d-1} - \frac{\omega_d}{2\pi}\kappa_{d-1}\frac{|\partial K|}{|K|}\int_0^t (1 - F_\sigma(s))\mathrm{d}s\right).$$

$$f_\Delta(x) = \frac{x^{d-1}}{|K|}\left(\omega_d - \frac{|\partial K|}{|K|}\kappa_{d-1}\int\limits_0^x (1 - F_\sigma(t))\,\mathrm{d}t\right).$$

Corollary

Corollary

Let K be a spherical cap of spherical radius $r < \frac{\pi}{2}$. Then

$$f_\Delta(t) = \omega_{d-1} \frac{\sin^{d-2} t}{|K|} \left(1 - \frac{\omega_d}{2\pi} \kappa_{d-1} \frac{1}{|K|} \int_0^t \left(1 - \frac{\cos^2 r}{\cos^2 \frac{s}{2}} \right)^{\frac{d-2}{2}} \mathrm{d}s \right).$$

Corollary

Corollary

If $d = 2m + 2$, *we have*

$$f_\Delta(t) = \omega_{d-1}\frac{\sin^{d-2} t}{|K|} - \frac{\omega_d\omega_{d-1}\kappa_{d-1}}{\pi}\frac{\sin^{d-2} t}{|K|^2}\tan\frac{t}{2}$$
$$\times \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{(2k-2)!!}{(2k-1)!!}(\cos r)^{2(m-k)}$$
$$\times \left(1 + \sum_{l=1}^{m-k-1}\frac{(2l-1)!!}{(2l)!!}\frac{1}{\cos^{2l}\frac{t}{2}}\right).$$

Weighted Gaussian simplex

- X — standard Gaussian vector in $\mathbb{R}^k$

$$|X| \stackrel{d}{=} \chi_k.$$

- $X_1, \dots, X_l$ — independent standard Gaussian vector in $\mathbb{R}^d$ $(l \leqslant d)$

$$|\mathrm{conv}(0, X_1, \dots, X_l)| \stackrel{d}{=} \frac{1}{l!}\chi_{d-l+1} \dots \chi_d.$$

- $\sigma_1, \dots, \sigma_l > 0$

$$|\mathrm{conv}(0, \sigma_1 X_1, \dots, \sigma_l X_l)| \stackrel{d}{=} \sigma_1 \dots \sigma_l \frac{1}{l!}\chi_{d-l+1} \dots \chi_d.$$

Known results

Theorem (R.E. Miles, 1971)

$X_0, \dots, X_l$ — *independent standard Gaussian vector in* $\mathbb{R}^d$.

$$\mathbb{E}|\mathrm{conv}(X_0, \dots, X_l)|^p = \left[\frac{2^{l/2}\sqrt{l+1}}{l!}\right]^p \prod_{i=d-l+1}^{d} \frac{\Gamma((i+p)/2)}{\Gamma(i/2)}.$$

$$\mathbb{E}\chi_k^p = 2^{p/2}\frac{\Gamma((k+p)/2)}{\Gamma(k/2)}$$

Theorem (R.E. Miles, 1971)

$$|\mathrm{conv}(X_0, \dots, X_l)| \stackrel{d}{=} \frac{\sqrt{l+1}}{l!}\chi_{d-l+1} \dots \chi_d.$$

Known results

Theorem (J. Randon-Furling, D. Zaporozhets, 2020)

$$\mathbb{E}|\mathrm{conv}(\sigma_0 X_0, \dots, \sigma_l X_l)|^p =$$
$$= \left[\frac{\sigma_0 \dots \sigma_l}{l!}\sqrt{\frac{1}{\sigma_0^2} + \dots + \frac{1}{\sigma_l^2}}\right]^p \prod_{i=d-l+1}^{d} \mathbb{E}\chi_i^p.$$

Main result

Theorem (T.M., 2020)

Fix some $l = 1, \dots, d$. Let $X_0, \dots, X_l$ be independent d-dimensional standard Gaussian vectors. Then for any $\sigma_0, \dots, \sigma_l > 0$

$$|\mathrm{conv}(\sigma_0 X_0, \dots, \sigma_l X_l)| \overset{d}{=} \frac{1}{l!}\sigma_0 \dots \sigma_l \sqrt{\frac{1}{\sigma_0^2} + \dots + \frac{1}{\sigma_l^2}}\chi_{d-l+1} \dots \chi_d,$$

where $\chi_{d-l+1}, \dots, \chi_d$ are independent random variables such that for any $k = d - l + 1, \dots, d$ the random variable χ_k has the chi distribution with k degrees of freedom.

Important Step

$$V_l := \mathrm{aff}(\sigma_0 X_0, \ldots, \sigma_l X_l), \quad l < d.$$

V_l — affine l-plane.
O_{V_l} — the orthogonal projection of the origin onto V_l.

$$W_l := V_l - O_{V_l}.$$

Lemma

W_l is uniformly distributed over the l-dimensional linear Grassmanian with respect to Haar measure and independently of $|\mathrm{conv}(\sigma_0 X_0, \ldots, \sigma_l X_l)|$.

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