On the convergence of the spectral radius of a random matrix with i.i.d. coefficients

(joint work with Charles Bordenave and Djalil Chafaï)

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Outline

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Girko matrix

$(a_{ij})_{i,j\geq 1}$: centered independent and identically distributed.

Girko matrix

$$A_n = (a_{ij})_{1\leq i,j\leq n}.$$

Characteristic polynomial :

$$P_n(z) = \det\left(z\mathbb{1}_n - A_n\right).$$

Eigenvalues :

$$\{z \in \mathbb{C} : P_n(z) = 0\}.$$

Examples : n = 500

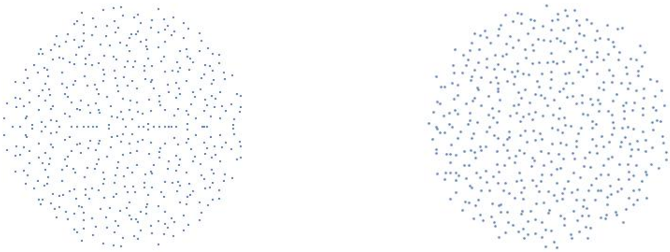

FIGURE: Rademacher and complex Gaussian

- $\mathbb{D}$: centered unit disk of $\mathbb{C}$,
- $\ell_{\mathbb{C}}$: Lebesgue measure of $\mathbb{C}$.

If $\mathbb{E}[|a_{11}|^2] = 1$ then, for $f : \mathbb{C} \to \mathbb{R}$ bounded continuous,

$$\frac{1}{n} \sum_{P_n(z)=0} f\left(\frac{z}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{\text{a.s.}} \frac{1}{\pi} \int_{\mathbb{D}} f(z) \mathrm{d}\ell_{\mathbb{C}}(z).$$

Girko (1984) . . . *Tao & Vu* (2010).

Weyl polynomial

$(a_i)_{i\geq 0}$: non-deterministic, independent and identically distributed.

Weyl polynomial

$$W_n(z) = \sum_{k=0}^{n} a_k \frac{z^k}{\sqrt{k!}}.$$

If $\mathbb{E}[\log(1+|a_0|)] < \infty$ then, for $f : \mathbb{C} \to \mathbb{R}$ bounded continuous,

$$\frac{1}{n} \sum_{W_n(z)=0} f\left(\frac{z}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{\text{a.s.}} \frac{1}{\pi} \int_{\mathbb{D}} f(z)\mathrm{d}\ell_{\mathbb{C}}(z).$$

Most general version by *Kabluchko & Zaporozhets* (2014).

Comparison : $n = 500$

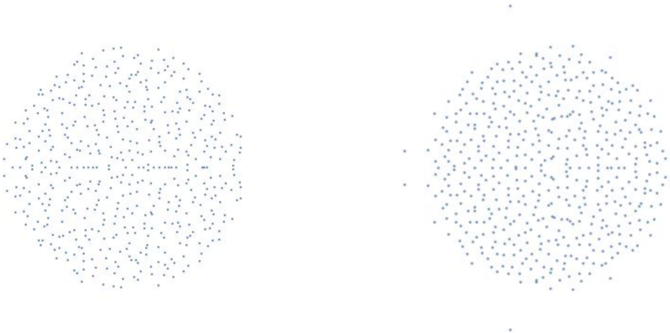

FIGURE: Girko matrix and Weyl polynomial

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Outliers for Weyl polynomials

Consider the random polynomial

$$\widetilde{W}_n(z) = \sqrt{n!}\left(\frac{z}{\sqrt{n}}\right)^n W_n\left(\frac{\sqrt{n}}{z}\right)$$

and the random holomorphic function $K : \mathbb{D} \to \mathbb{C}$

$$K(z) = \sum_{k=0}^{\infty} a_k z^k .$$

If $\mathbb{E}[\log(1 + |a_0|)] < \infty$ then [Butez & G-Z (2020)]

$$\widetilde{W}_n|_{\mathbb{D}} \xrightarrow[n\to\infty]{\text{law}} K .$$

This explains the outliers !

Lack of outliers for Girko matrices

$(X_k)_{k\geq 1}$: independent centered complex Gaussians such that

$$\mathbb{E}[|X_k|^2] = 1 \quad \text{and} \quad \mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k.$$

Let $\kappa : \mathbb{D} \to \mathbb{C}$ be the holomorphic function given by

$$\kappa(z) = \sqrt{1 - z^2\mathbb{E}[a_{11}^2]},$$

and $F : \mathbb{D} \to \mathbb{C}$, the random holomorphic function defined by

$$F(z) = \sum_{k=1}^{\infty} X_k \frac{z^k}{\sqrt{k}}.$$

Recall that

$$P_n(z) = \det\left(z\mathbb{1}_n - A_n\right)$$

and consider the random polynomial

$$\widetilde{P}_n(z) = \left(\frac{z}{\sqrt{n}}\right)^n P_n\left(\frac{\sqrt{n}}{z}\right) = \det\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right).$$

Theorem (Bordenave, Chafaï and G-Z, 2020)

If $\mathbb{E}[|a_{11}|^2] = 1$ *then*

$$\widetilde{P}_n|_{\mathbb{D}} \xrightarrow[n\to\infty]{\text{law}} \kappa e^{-F}.$$

This explains the lack of outliers !

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Exponential of traces

$$\det\left(e^{B}\right) = e^{\mathrm{Tr}\, B} \quad\Longrightarrow\quad \det\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right) = e^{\mathrm{Tr}\log\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right)}.$$

Since

$$\log\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right) = -\sum_{k=1}^{\infty}\left(\frac{A_n}{\sqrt{n}}\right)^k\frac{z^k}{k}$$

we obtain

$$\det\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right) = e^{-\sum_{k=1}^{\infty}\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^k\right]\frac{z^k}{k}}.$$

Convergence of traces

The idea is to show that

$$\left(\frac{1}{k}\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^k\right]\right)_{k\geq 1} \xrightarrow[n\to\infty]{\text{law}} \left(\frac{X_k}{\sqrt{k}} + \frac{\mathbb{E}[a_{11}^2]^{k/2}}{k}1_{k \text{ is even}}\right)_{k\geq 1},$$

where $(X_k)_{k\geq 1}$ are independent centered complex Gaussians,

$$\mathbb{E}[|X_k|^2] = 1 \quad \text{and} \quad \mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k.$$

If all the moments of a_{11} *are finite :* **combinatorial argument !**

In general : **truncation argument**.
Inspired by a work of *Svante Janson & Krzysztof Nowicki* (1991).

Tightness of $\widetilde{P}_n$

Write

$$\widetilde{P}_n(z) = \det\left(\mathbb{1}_n - z\frac{A_n}{\sqrt{n}}\right) = \sum_{k=0}^{n} z^k C_k^{(n)}$$

and notice that

$$\mathbb{E}\left[|C_k^{(n)}|^2\right] \leq 1 \quad \text{and} \quad \mathbb{E}\left[C_k^{(n)}\bar{C}_{k'}^{(n)}\right] = 0 \text{ if } k \neq k'.$$

Then,

$$\mathbb{E}\left[|\widetilde{P}_n(z)|^2\right] \leq \sum_{k=0}^{n} |z|^{2k} \leq \frac{1}{1-|z|^2}.$$

Inspired by a work of *Anirban Basak & Ofer Zeitouni* (2020).

- Janson and Nowicki. *The asymptotic distributions of generalized U-statistics with applications to random graphs.* Probability Theory and Related Fields **90** (1991), no. 3, 341–375.
- Basak and Zeitouni. *Outliers of random perturbations of Toeplitz matrices with finite symbols.* Probability Theory and Related Fields **178** (2020), no. 3-4, 771–826.
- Butez and G-Z. *Extremal particles of two-dimensional Coulomb gases and random polynomials on a positive background.* Preprint arXiv :1811.12225 (2020).
- Bordenave, Chafaï and G-Z *Convergence of the spectral radius of a random matrix through its characteristic polynomial.* Preprint arXiv :2012.05602 (2020).

Thank you for your attention !