Symmetric Lévy processes with reflection

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Overview

Reflecting processes

- Geometric reflection – c'est pas ça
- Reflection is Neumann conditions for the generator
- Local time huddles the process back into the domain
- Example of Skorokhod reflection

Construction

- A class of Lévy processes under consideration
- Main idea: two extensions
- Difference of semigroups lives on the boundary
- Group structure of our operator families
- Pathwise accumulated momentum
- Limit theorem for the reflecting BM in d-ball

Part 1. General notions, some intuition and historical notes

Geometric reflection – c'est pas ça

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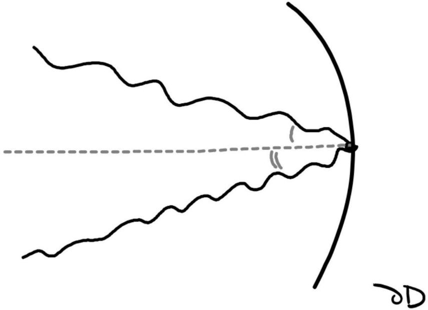
Geometric reflection – c'est pas ça

Let $\xi(t)$ be a Markov process in $\mathbb{R}^d$. We shall call it *free* in opposition to its *confined to D* versions. We are interested in one single type of containment: *specular reflection* off the boundary. Note that we cannot define the reflection geometrically



since it means imposing conditions on normal derivatives, which very seldom exist.

Reflection is Neumann conditions for the generator

Taking average of f over the paths of ξ

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is a Markov contraction semigroup! We *define* the reflecting version of ξ by imposing Neumann conditions on its generator. Or on its quadratic form.

Example (see. [1])

Brownian Motion with reflection off the boundary ∂D is a process, associated with the Dirichlet form

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \nabla u \cdot \overline{\nabla v} \, d\mathbf{x}, \quad D[\mathcal{E}] = W_2^1(D).$$

Remark. Same form on $W_2^{1,0}(D)$ gives rise to an *absorbing* BM.

Local time huddles the process back into the domain

Skorokhod '61: $d = 1$

Skorokhod in [6] has proposed a *pathwise* construction of the reflecting diffusion in $[0, \infty)$ and proved the since-called Skorokhod semimartingale decomposition

$$\widetilde{\xi}(t) = \xi(t) + \zeta(t),$$

where ζ is the local time of ξ.

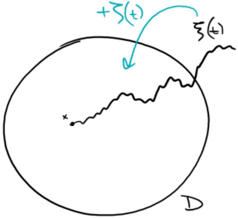

Figure: Local time pushes the process back into the domain D

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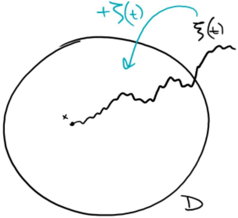
Case $d > 1$: $d\zeta$ is a measure on a boundary

If the boundary is C^3-smooth, the Skorokhod construction is still applicable (see [3]), but in place of $\zeta(t)$ the decomposition features an integral with respect to $d\zeta(t)$.



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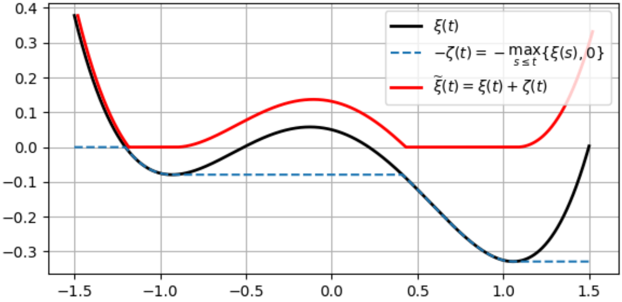
Skorokhod reflection bears no resemblance to the geometric reflection

Let ξ be a *non-random* smooth function. Its reflecting version in $[0, \infty)$:



Here $\widetilde{\xi}$ is the reflecting version and ζ is the compensator.

Part 2. Construction

A class of Lévy processes under consideration

consists of

1. pure jump processes (without diffusion terms)
2. with rotation-invariant Lévy measure $d\Pi$
3. and finite second moments

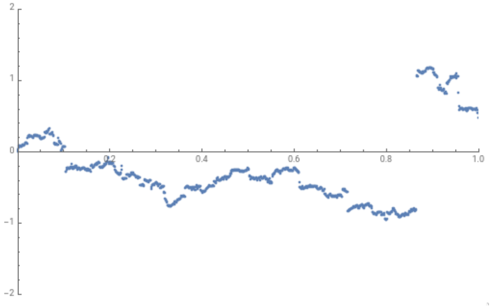


Figure: Typical path of α-stable process ($\alpha = 1.42$),
`https://demonstrations.wolfram.com/StableLevyProcess`

Characteristic function and generator

The characteristic function

is of the form $\varphi_t(\mathbf{p}) = \exp(-tL(\mathbf{p}))$, where

$$L(\mathbf{p}) = -\int_{\mathbb{R}^d} \left(e^{i\mathbf{p}\cdot\mathbf{x}} - 1 - i\mathbf{p}\cdot\mathbf{x}\right) d\Pi(\mathbf{x})\,. \qquad (1)$$

Generator

$$-Lf(\mathbf{x}) = \int_{\mathbb{R}^d} \Big(f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - f'(\mathbf{x})\cdot\mathbf{y}\Big)\, d\Pi(\mathbf{y}) \qquad (2)$$

Remark. It *would* be a PDO if $L(\mathbf{p})$ was smooth (it in general is not!).

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1. take a function f in D
2. and extend it in some special way to the whole $\mathbb{R}^d$:
   2.1 $\widetilde{f}$ for the reflecting process
   2.2 $\overline{f}$ for the process inside the domain
3. associate a semigroup to each by

$$P^t f = T^t \widetilde{f}, \quad R^t f = T^t \overline{f},$$

denote its generators by $-A$ and $-A^N$.

Adjustment to the main idea

The idea works well when D is very simple (such as a ball or a cube). In an arbitrary smooth domain we construct instead two sequences $\widetilde{f}_M(\mathbf{x},\mathbf{y})$ and $\overline{f}_M(\mathbf{x},\mathbf{y})$, which are tangent to f at $\mathbf{y}=0$.

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Then, we define P^t as

$$(P^t f)(\mathbf{x}) = \lim_{M\to\infty} \mathbb{E}\widetilde{f}_M(\mathbf{x},\xi(t)),$$

whereas for the generator we have

$$(Af)(\mathbf{x}) = \lim_{M\to\infty} L_{\mathbf{y}}\widetilde{f}_M(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=0}.$$

Difference lives on the boundary

First lemma on the difference of semigroups

Let $f \in W_2^2(D)$ and $\mathbf{x} \in D$. Then

$$(P^t f)(\mathbf{x}) - (R^t f)(\mathbf{x}) = -\lim_{M \to \infty} \int_0^t P^\tau L\Big(\widetilde{f}_M - \overline{f}_M\Big)(\mathbf{x}, 0)\, d\tau$$

and in $W_2^2(D)$.

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Remark. We shall employ this lemma to define a *pathwise* accumulated momentum.

Difference of semigroups lives on the boundary

Second lemma on the difference of semigroups

For every $f \in W_2^2(D)$ holds

$$(P^t f)(\mathbf{x}) - (R^t f)(\mathbf{x}) = \int_{\partial D} Q^t(\mathbf{x}, \mathbf{z})(\gamma_1 f)(\mathbf{z})\, dS(\mathbf{z}),$$

where

$$Q^t(\mathbf{x}, \mathbf{z}) = \frac{1}{2}\int_0^t \widetilde{R}^\tau(\mathbf{x}, \mathbf{z})\, d\tau, \quad \widetilde{R}^\tau(\mathbf{x}, \mathbf{z}) = \sum_{l=0}^{\infty} \frac{L(\kappa_l)}{\kappa_l^2} e^{-tL(\kappa_l)} s_l(\mathbf{x})\overline{s_l(\mathbf{z})}.$$

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Second lemma on the difference of semigroups

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Remark. Recall that local time is a measure on the boundary.
It is exactly the case here: $d\mathcal{L}_\tau = \frac{1}{2} \widetilde{R}^\tau(\mathbf{x}, \mathbf{z}) \delta_{\partial D}(\mathbf{z})\, d\tau$.

A small digression regarding Brosamler theorem

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Classical result due to Brosamler [2]:

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In our setting,

$$u(t, \mathbf{x}) = (Q^t g)(\mathbf{x})$$

solves $-Au = 0$, $u(0, \mathbf{x}) = 0$, $\gamma_1 u = g$.

Three corollaries on the group structure (1)

Theorem 1

The operator families $(R^t)_{t\geq 0}$ and $(Q^t)_{t\geq 0}$ satisfy the following evolution relations:

$$
\begin{aligned}
R^{t+s} &= R^t R^s, \\
Q^{t+s} &= Q^t + \widetilde{R}^t Q^s
\end{aligned}
$$

and $R^0 = I$, $Q^0 = 0$.

Three corollaries on the group structure (2)

Theorem 2

For every $t > 0$ and $f \in L_2(D)$ holds

$$\frac{\partial}{\partial t} R^t f = \frac{1}{2} A^N R^t f.$$

Three corollaries on the group structure (2)

Theorem 2

For every $t > 0$ and $f \in L_2(D)$ holds

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Theorem 3

For every $t > 0$ and $g \in W_2^{1/2}(\partial D)$ holds

$$\frac{\partial}{\partial t} Q^t g = \frac{1}{2} \int_{\partial D} \widetilde{R}^t(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) \, dS(\mathbf{y}) \, .$$

Pathwise definition of the accumulated momentum

It is easy to guess a pathwise counterpart of P^t – is should be a shift by $\xi(t)$

$$(\mathcal{P}^t f)(\mathbf{x}) = f(\mathbf{x} + \xi(t))$$

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By way of analogy with the first lemma, define:

$$(\mathcal{Q}^t g)(\mathbf{x}) = \lim_{M\to\infty} \int_0^t \mathcal{P}^\tau L\Big(\widetilde{G}_M - \overline{G}_M\Big)(\mathbf{x}, 0)\, d\tau, \qquad (3)$$

where $G(\mathbf{x}, \mathbf{y})$ is a special continuation of g into the domain.

Properties of pathwise accumulated momentum

Existence

The limit in the rhs of (3) exists in $L_2(\mathcal{H}, \mu)$, where $\mathcal{H} = D \times \Omega$ and $d\mu = d\mathbf{x} \times d\mathbf{P}$.

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The average of $\mathcal{Q}^t$

Let $g \in W_2^{1/2}(\partial D)$. Then

$$\mathbb{E}(\mathcal{Q}^t g)(\mathbf{x}) = (Q^t g)(\mathbf{x}).$$

Limit theorem for the reflecting BM in d-ball

([5] by me). Let us approximate the BM $w(t)$ with a random walk

$$\zeta_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\eta(nt)} \xi_j.$$

Here ξ_j are iidrvs with common rotation-invariant distribution and a unitary second moment.

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From this we derive the definition of Q_n^t.

Limit theorem for the reflecting BM in d-ball

Limit theorem for the process in the domain

Let $f \in D(A^N)$. Then

$$\left\| R_n^t f - R^t f \right\|_{L_2(D)} \leq \frac{C\sqrt{t}}{\sqrt{n}} \left\| f \right\|_{W_2^2(D)} .$$

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Limit theorem for the local time

For every $g \in W_2^{1/2}(\partial D)$ holds

$$\left\| Q_n^t g - Q^t g \right\|_{L_2(D)} \le \frac{Ct^{3/8}}{n^{3/8}} \|g\|^2_{W_2^{1/2}(\partial D)}.$$

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