Global stability in a stochastic market model

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Introduction

This talk is based on the paper arXiv:2008.13230 (see also 1811.12491, 1908.01171, 2007.04909).

We consider a market model with N agents who can trade M assets which yield random payoffs.

The main result consists in constructing a strategy which cannot be driven out of the market, and which drives all other essentially different strategies out of the market .

Evolutionary ideas in economics

- Alchian (1950)

  > "The economics counterparts of genetic heredity, mutations, and natural selection are imitation, innovation, and positive profits."

- Friedman (1953)

  > "Whenever this determinant [of business behavior] happens to lead to behavior consistent with rational and informed maximization of returns, the business will prosper and acquire resources with which to expand; whenever it does not, the business will tend to lose resources and can be kept in existence only by the addition of resources from outside."

Some mathematical models

- De Long et al. (1990, 1991): noise traders can outperform rational agents.
- Blume, Easley (1992): market forces in a complete market select for agents depending on accuracy of their beliefs and (exogenous) savings rates.
- Sandroni (2000), Blume, Easley (2006): in a complete market with endogenous saving rates, only accuracy of beliefs is important.
- Yan (2008), Dindo (2015), Borovicka (2020): further results on selection in complete markets.
- Evstigneev et al. (2002), Hens, Schenk-Hoppe (2005): selection and stability in incomplete markets.

This paper

The papers extends the results of Amir et al. (2013) and Drokin and Zhitlukhin (2020) to continuous time:

- a continuous-time model,
- the market may be incomplete,
- agents can use arbitrary strategies (not necessarily utility maximization),
- a proof of existence (and an explicit construction) of a strategy which asymptotically outperforms other strategies.

The model

- A filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$ satisfying the usual assumptions.
- N assets with cumulative payoff processes $X_t^n(\omega)$, which are adapted to $\mathscr{F}_t$, non-decreasing, and right-continuous.
- M agents with cumulative investment processes $L_t^m(\omega) \in \mathbb{R}_+^N$ of the form

$$L_t^m(\omega) = \int_0^t l_s^m(\omega, Y_{s-}(\omega))ds,$$

where $l_t^m = (l_t^{m,1}, \ldots, l_t^{m,n}) \in \mathbb{R}_+^N$ are predictable[1] investment intensities, and $Y_s \in \mathbb{R}_+^M$ is the process of agents wealth, which is yet to be defined.

[1]Measurable with respect to $\mathcal{P} \otimes \mathscr{B}(\mathbb{R}_+^M)$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbb{R}_+$.

Wealth equation

The vector process Y_t is defined by the equation (ω is omitted for brevity)

$$dY_t^m = \sum_{n=1}^{N} \left(-l_t^{m,n}(Y_{t-})dt + \frac{l_t^{m,n}(Y_{t-})}{\sum\limits_{k=1}^{M} l_t^{k,n}(Y_{t-})} dX_t^n \right).$$

In this model, the assets are short-lived – they cannot be held and sold later.

Assumption on X_t

Let $\widetilde{X}_t = X_t - \sum_{s\leqslant t} \Delta X_s$ be the continuous part of X_t, and ν_t be the compensator of its measure of jumps. We will assume that the "operational time process"

$$G_t = \|\widetilde{X}_t\| + (1 \wedge \|x\|) * \nu_t,$$

is absolutely continuous, $G_t = \int_0^t g_s ds$ with $g_s \geqslant 0$.

Remark: a more general model can be considered

We can actually assume that

- X_t is an arbitrary non-decreasing (in each coordinate) right-continuous adapted cumulative payoff process,
- a strategy profile $(L_t^1, \dots, L_t^M)$ consists of arbitrary non-decreasing right-continuous predictable processes $L_t^m(\omega, \bar{Y})$ which can depend on the whole history of the wealth process $\bar{Y} = (Y_s,\ s < t)$, such that $Y_t \in \mathbb{R}_+^m$,
- $l_t^m(\omega)$ are derivatives of $L_t^m(\omega, \bar{Y}(\omega))$ w.r.t. appropriate dominating process.

The existence of a unique solution of the wealth equation can be proved under continuity and boundedness assumptions on l_t^m (see the paper).

Relative wealth: survival, dominance, stability

Define the total wealth of all agents and the relative wealth of agent m:

$$W_t = \sum_{m=1}^{M} Y_t^m, \qquad R_t^m = \frac{Y_t^m}{W_t}.$$

Call a strategy L of agent m

- survival, if $P(\inf_{t \geqslant 0} R_t^m > 0) = 1$ in any strategy profile,
- dominating in a given strategy profile, if $P(\lim_{t \to \infty} R_t^m = 1) = 1$.

If L is a dominating strategy for a given strategy profile and any initial wealth $Y_0 \in \mathbb{R}_+^M$ with $Y_0^m > 0$, then $(0, \ldots, 0, 1, 0, \ldots, 0)$ is a globally stable state of the random dynamical system $(R_t^1, \ldots, R_t^M)$.

Construction of a survival strategy

Auxiliary objects

Split the payoff process $X_t = (X_t^1, \dots, X_t^N)$ into the continuous part $\widetilde{X}_t$ and the sum of jumps:

$$X_t = \widetilde{X}_t + \sum_{\substack{s \leqslant t \\ \Delta X_s \neq 0}} \Delta X_s.$$

Let $\mu(\omega, dt, dx)$ be the measure of jumps of X_t:

$$\mu(\omega, A) = \sum_{\substack{t \geqslant 0 \\ \Delta X_t \neq 0}} I((t, \Delta X_t(\omega)) \in A), \qquad A \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}_+^N).$$

Auxiliary objects (continued)

Let $\nu(\omega, dt, dx)$ be the compensator of μ, i.e. the random measure such that

(a) the process $f * \nu_t$ is predictable,

(b) $E(f * \mu_t) = E(f * \nu_t)$

for any function $f(\omega, s, x) \geqslant 0$ measurable w.r.t. $\mathcal{P} \otimes \mathscr{B}(\mathbb{R}_+^N)$, where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbb{R}_+$ (generated by all continuous adapted processes).

Here and below the star $*$ denotes integration, i.e.

$$f * \mu_t(\omega) = \int_0^t f(\omega, ds, dx) \mu(\omega, ds, dx)$$

According to our assumptions $G_t := \|\widetilde{X}_t\| + (1 \wedge \|x\|) * \nu_t = \int_0^t g_s ds$, which implies the existence of a predictable processes $b_t(\omega)$ with values in $\mathbb{R}_+^N$ and a transition kernel $K_{\omega,t}(dx)$ such that

$$\widetilde{X}_t = \int_0^t b_s ds, \qquad \nu(\omega, dt, dx) = K_{\omega,t}(dx) dt.$$

Theorem 1

Define the $\mathcal{P} \otimes \mathscr{B}(\mathbb{R}_+)$-measurable function $\widehat{\lambda}(\omega, t, c)$ with values in $\mathbb{R}_+^N$ and the components

$$\widehat{\lambda}_t^n(c) = \frac{b_t^n}{c} + \int_{\mathbb{R}_+^N} \frac{x^n}{c + |x|} K_t(dx)$$

(where $|x| = \sum_n |x^n|$ for $x \in \mathbb{R}^N$).

Then the following strategy (of agent m) is survival:

$$\widehat{L}_t^n = \int_0^t Y_{s-}^m \widehat{\lambda}_s^n(W_{s-}) ds = \int_0^t R_{s-}^m d\widetilde{X}_s^n + \left(\frac{x^n Y_-^m}{W_- + |x|} \right) * \nu_t.$$

Theorem 2

Suppose agent $m = 1$ uses the strategy $\widehat{L}$. Denote by $\widetilde{\lambda}_t$ the investment proportion of the representative agent of agents $m = 2, \dots, M$:

$$\widetilde{\lambda}_t^n = \sum_{m=2}^{M} \frac{r_{t-}^m}{1 - r_{t-}^1} \lambda_t^{m,n}, \qquad \text{where } \lambda_t^{m,n}(\omega) = \frac{l_t^{m,n}(\omega, Y_{t-}(\omega))}{Y_{t-}^m(\omega)}$$

Then

$$r_t^1 \to 1 \text{ a.s. on the set } \left\{ \omega : \int_0^\infty \| \lambda_s^1(\omega) - \widetilde{\lambda}_s(\omega) \|^2 ds = \infty \right\}.$$

Main steps of the proofs

Proof of Theorem 1

The main step is to prove that if agent $m = 1$ uses the strategy $\widehat{L}$, then

$$Z_t = \ln R_t^1 \text{ is a submartingale,}$$

i.e. $E|Z_t| < \infty$ and $E(Z_t \mid \mathscr{F}_s) \geqslant Z_s$ for any $0 \leqslant s < t$. Since a non-positive submartingale has a finite limit, there exists $\rho = \lim_{t\to\infty} Z_t$ and $\lim_{t\to\infty} r_t^1 = e^\rho > 0$.

It will be enough to prove that Z_t is a σ-submartingale, i.e. there exists a sequence of predictable sets $\Pi_n \in \mathcal{P}$ such that $Z_t^{\Pi_n} := \int_0^t \mathbf{I}_s(\Pi_n) dZ_s$ is a submartingale for each n and $\bigcup_n \Pi_n = \Omega \times \mathbb{R}_+$ (since a non-positive σ-submartingale is a submartingale).

A sufficient condition for σ-submartingality in terms of its drift coefficient

Suppose S is a semimartingale with a triplet of predictable characteristics (C, B, ν) with respect to some truncation function $h(x)$ (e.g. $h(x) = x\mathbf{I}(|x| \leqslant 1)$), so

$$S_t = S_0 + S_t^c + B_t + h(x) * (\mu - \nu)_t + (x - h(x)) * \mu_t, \qquad C_t = \langle S^c \rangle_t.$$

It is always possible to find a predictable non-decreasing process A_t such that

$$C_t = \int_0^t c_s dA_s, \qquad B_t = \int_0^t b_s dA_s, \qquad \nu(\omega, dt, dx) = K_{\omega,t}(dx) dA_t.$$

Proposition (Kallsen 2003): Z is a σ-submartingale if and only if $(P \otimes A)$-a.s.

$$\int_{|x|>1} |x| K_t(dx) < \infty \quad \text{and} \quad \mathfrak{d}_t := b_t + \int_{\mathbb{R}} (x - h(x)) K_t(dx) \geqslant 0,$$

where the predictable process $\mathfrak{d}_t(\omega)$ is called the drift coefficient.

Corollary: a non-positive semimartingale is a σ-submartingale (and, hence, a submartingale) if $P \otimes A$-a.s.

$$\int_{x<0} x K_t(dx) > -\infty \qquad \text{and} \qquad \mathfrak{d}_t = b_t^0 + \int_{\mathbb{R}} x K_t(dx) \geqslant 0,$$

where $b_t^0 = b_t - \int_{\mathbb{R}} h(x) K_t(dx)$ is a well-defined predictable process (the same for any truncation function).

We compute that for our process $Z_t = \ln R_t^1$

$$\mathfrak{d}_t = (1 - R_{t-}^1)(|\widetilde{\lambda}_t| - |\widehat{\lambda}_t|) + \frac{(F_t - 1) \cdot b_t}{W_{t-}} + \int_{\mathbb{R}_+^N} f_t(x) K_t(dx),$$

where

$$F_t^n = \frac{\widehat{\lambda}_t^n}{R_{t-}^1 \widehat{\lambda}_t^n + (1 - R_{t-}^1)\widetilde{\lambda}_t^n}, \qquad f_t(x) = \ln\left(\frac{W_{t-} + F_t \cdot x}{W_{t-} + |x|}\right).$$

Using that

$$(F_t - 1) \cdot b_t \geqslant \ln(F_t) \cdot b_t, \qquad f_t(x) \geqslant \frac{x \cdot \ln F_t}{W_{t-} + |x|},$$

we find

$$\mathfrak{d}_t \geqslant (1 - R_{t-}^1)(|\widetilde{\lambda}_t| - |\widehat{\lambda}_t|) + \frac{(a_t + b_t) \cdot \ln F_t}{W_{t-}} \quad \text{with } a_t = \int_{\mathbb{R}_+^N} \frac{x}{W_{t-} + |x|} K_t(dx).$$

Using that $\widehat{\lambda}_t = (a_t + b_t)/W_{t-}$, we find

$$\begin{aligned}
\mathfrak{d}_t &\geqslant (1 - R^1_{t-})(|\widetilde{\lambda}_t| - |\widehat{\lambda}_t|) + \widehat{\lambda}_t \cdot \ln F_t \\
&= (1 - R^1_{t-})(|\widetilde{\lambda}_t| - |\widehat{\lambda}_t|) + \widehat{\lambda}_t \cdot (\ln \widehat{\lambda}_t - \ln(R^1_{t-}\widehat{\lambda}_t + (1 - R^1_{t-})\widetilde{\lambda}_t)) \\
&\geqslant \frac{1}{4}(1 - R^1_{t-})^2 \|\widehat{\lambda}_t - \widetilde{\lambda}_t\|^2 \geqslant 0. \qquad (*)
\end{aligned}$$

Proof of Theorem 2

Since Z_t is a convergent submartingale, its compensator A_t also converges, i.e. $\lim_{t\to\infty} A_t < \infty$, where

$$A_t = \int_0^t \mathfrak{d}_s ds.$$

From $(*)$, we find that if $\lim_{t\to\infty} R^1_t < \infty$, then necessarily

$$\int_0^\infty \|\widehat{\lambda}_t - \widetilde{\lambda}_t\|^2 < \infty.$$

Thank you for your attention

Have a happy and healthy New Year!