A law of large numbers for local patterns in random plane partitions

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- Presentation of the model

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- The limit processes

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- The proof

Presentation of the model

A plane partition $\pi = (\pi_{i,j})_{i,j \geq 1}$ is an almost zero double array of non-negative integers, which is non-increasing in both directions :

$$\pi_{i,j} \geq \pi_{i+1,j} \quad \text{and} \quad \pi_{i,j} \geq \pi_{i,j+1},$$

$$|\pi| := \sum_{i,j=1}^{+\infty} \pi_{i,j} < +\infty.$$

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We equip the set of plane partition with a geometric weight :

$$\mathbb{P}_q(\pi) = C.q^{|\pi|},$$

where $q \in (0,1)$ and C is the normalizing contant.

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where $q \in (0,1)$ and C is the normalizing contant. We associate to a plane partition π a configuration on $E := \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$:

$$\mathfrak{S}(\pi) = \{(i-j), \pi_{i,j} - (i+j-1)/2,\ i,j \geq 1\}.$$

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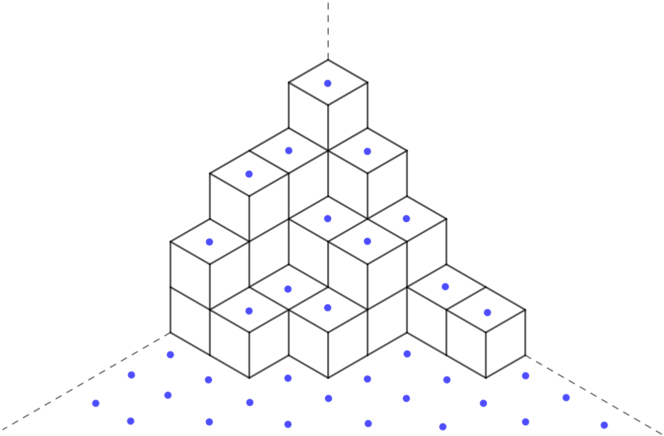

Figure – The plane partition $\begin{pmatrix} 4 & 3 & 2 & 1 & 1 \\ 3 & 2 & 2 & & \\ 3 & 1 & 1 & & \\ 2 & 1 & & & \end{pmatrix}$ and its associated configuration

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Theorem (Okounkov-Reshetikhin, 2003)

The image of $\mathbb{P}_q$ *by* $\mathfrak{S}$ *is a determinantal process with kernel :*

$$K_q(s,x;t,y) = \frac{1}{(2i\pi)^2} \int_{|z|=1\pm\epsilon} \int_{|w|=\mp\epsilon} \frac{\Phi(s,z)}{\Phi(t,w)} \frac{1}{z-w} \frac{dzdw}{z^{x+\frac{|s|+1}{2}} w^{-y-\frac{|t|-1}{2}}}.$$

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- *The number $\epsilon > 0$ is sufficiently small in order to count singularities.*
- *One takes the sign "$+$" for $s \geq t$ and the sign "$-$" for $s < t$.*

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where : $\{z(\tau,\chi), \overline{z(\tau,\chi)}\} = C(0, e^{-\tau/2}) \cap C(1, e^{-\tau/4-\chi/2})$.
Then, for all $(\tau,\chi) \in A$, *and all* $(t_1,h_1), \ldots, (t_n,h_n) \in E$, *we have :*

$$\lim_{r \to 0^+} \det \left(K_{e^{-r}} \left(\frac{\tau}{r} + t_i, \frac{\chi}{r} + h_i; \frac{\tau}{r} + t_j, \frac{\chi}{r} h_j \right) \right)_{i,j=1}^n = \det \left(\mathcal{S}_{\tau,\chi}(t_i - t_j, h_i - h_j) \right)_{i,j=1}^n .$$

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For $m = \left(\left(m_1^1, m_1^2 \right), ..., \left(m_n^1, m_n^2 \right) \right) \subset E$, we write :

$$\mathbb{1}_m(\pi) = 1 \text{ if } m \subset \mathfrak{S}(\pi), \text{ 0 else.}$$

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$$\lim_{r \to 0} \mathbb{P}_{e^{-r}} \left(\left| r^2 \sum_{(t,h) \in E \cap r^{-1}A} f(rt, rh) \mathbb{1}_{(t,h)+m} - \int_A f(\tau, \chi) \mathbb{E}_{\tau,\chi}[\mathbb{1}_m] d\tau d\chi \right| > \varepsilon \right)$$

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We will prove that :

- the expectation of the left-hand-side converges to the right-hand-side,
- the variance of the left-hand-side goes to 0.

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is a Riemann sum for the integral :

$$\int_A f(\tau, \chi) \mathbb{E}_{\tau,\chi}[\mathbb{1}_m] d\tau d\chi.$$

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We prove that the variance :

$$r^4 \sum_{(t_1 h_1),(t_2,h_2)\in r^{-1}A\cap E} f(rt_1, rh_1) f(rt_2, rh_2)$$
$$\times \left(\mathbb{E}_r\left[\mathbb{1}_{(t_1,h_1)+m}\mathbb{1}_{(t_2,h_2)+m}\right] - \mathbb{E}_r\left[\mathbb{1}_{(t_1,h_1)+m}\right]\mathbb{E}_r\left[\mathbb{1}_{(t_2,h_2)+m}\right]\right)$$

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goes to 0 as r goes to 0. To this aim, we control the covariances :

$$\mathbb{E}_r\left[\mathbf{1}_{(t_1,h_1)+m}\mathbf{1}_{(t_2,h_2)+m}\right] - \mathbb{E}_r\left[\mathbf{1}_{(t_1,h_1)+m}\right]\mathbb{E}_r\left[\mathbf{1}_{(t_2,h_2)+m}\right]$$

in a convenient manner, depending on the fact that (t_1, h_1) and (t_2, h_2) are close of each other or not.

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$$\left|\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right] - \mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\right]\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right]\right| \leq \frac{C\exp\left(-r^{-\alpha}\right)}{|\tau_1 - \tau_2|^2}$$

Main lemma : control of the covariances

Lemma

For all finite subset $m \subset E$, for all compact set $\mathcal{K} \subset \mathbb{R}^2$ and for all $\alpha \in (0,1)$, there exists $C > 0$ such that for all sufficiently small $r > 0$, if (τ_1, χ_1), $(\tau_2, \chi_2) \in \mathcal{K} \cap A \cap rE$ are such that : $\max\{|\tau_1 - \tau_2|, |\chi_1 - \chi_2|\} > r||m||_\infty$, then, for $\tau_1 \neq \tau_2$:

$$\left|\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right] - \mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\right]\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right]\right| \leq \frac{C\exp\left(-r^{-\alpha}\right)}{|\tau_1 - \tau_2|^2}$$

and for $\tau_1 = \tau_2 = \tau$:

$$\left|\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau,\chi_1)+m}\mathbf{1}_{\frac{1}{r}(\tau,\chi_2)+m}\right] - \mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau,\chi_1)+m}\right]\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau,\chi_2)+m}\right]\right| \leq \frac{Cr}{|\chi_1 - \chi_2|}.$$

Proof of the limit Theorem

Since :

$$\log\left(\Phi(\tau/r+t,z)z^{-\chi/r-h-\frac{\tau/r+s+1}{2}}\right)\sim\frac{1}{r}S(z;\tau,\chi),$$

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the kernel $K_{e^{-r}}$, taken at $\frac{1}{r}(\tau,\chi)+(t_i,h_i)$, is equivalent, as $r\to 0$, to :

$$\int_{z\in(1\pm\epsilon)\gamma_\tau}\int_{w\in(1\mp\epsilon)\gamma_\tau}\exp\left(\frac{1}{r}\left(S(z;\tau,\chi)-S(w;\tau,\chi)\right)\right)/(z-w)dzdw,$$

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where :

- $\gamma_\tau=C(0,e^{\tau/2})$;

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where :

- $\gamma_\tau=C(0,e^{\tau/2})$;
- $\Re S(z;\tau,\chi)\equiv-\tau/2(\tau/2+\chi)$, $z\in\gamma_\tau$;
- when $(\tau,\chi)\in A$, the function S has two distinct critical points : $e^{\tau/2}z(\tau,\chi)$ and $e^{\tau/2}\overline{z(\tau,\chi)}$;

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- when $(\tau,\chi)\in A$, the function S has two distinct critical points : $e^{\tau/2}z(\tau,\chi)$ and $e^{\tau/2}\overline{z(\tau,\chi)}$;
- we can deform the contour γ_τ following the direction of the gradient of $\Re S(z)$.

Deformation of the contour

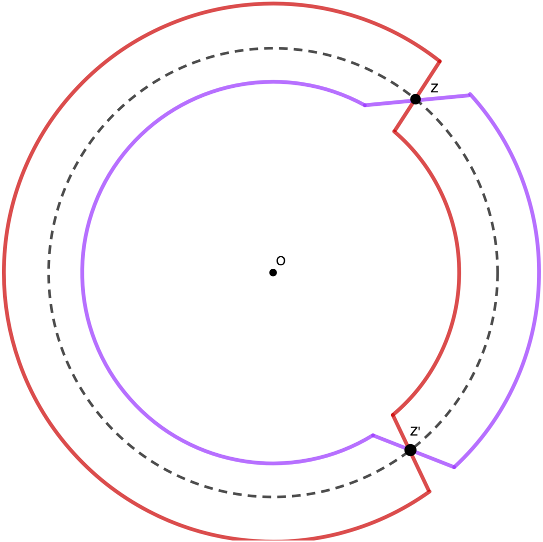

Figure – The contours $\gamma_\tau^>$ et $\gamma_\tau^<$

Deformation of the contour

We deform the circle γ_τ in two simple contours $\gamma_\tau^>$ et $\gamma_\tau^<$, following the direction (or the opposite direction) of the gradient of $\Re S(z;\tau,\chi)$ in such a way that :

$$\Re S(z;\tau,\chi) - \Re S(w;\tau,\chi) < 0,\ z \in \gamma_\tau^<,\ w \in \gamma_\tau^>,$$

except at the critical points where we have equality.

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except at the critical points where we have equality. Taking into account the residue at $z = w$, which we do not integrate in the new contours, one obtains the Theorem. $\square$

Proof of the main lemma

When we have :

$$\max\{|\tau_1 - \tau_2|, |\chi_1 - \chi_2|\} > r||m||_\infty,$$

the ensembles :

$$\frac{1}{r}(\tau_1, \chi_1) + m \text{ et } \frac{1}{r}(\tau_2, \chi_2) + m$$

are disjoint.

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are disjoint. This implies :

- The covariance :

$$\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right] - \mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\right]\mathbb{E}_r\left[\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right],$$

  can be written as a sum in which each term has a factor :

$$K_{e^{-r}}\left(\frac{\tau_1}{r} + t_1^1, \frac{\chi_1}{r} + h_1^1; \frac{\tau_2}{r} + t_2^1, \frac{\chi_2}{r} + h_2^1\right) \times K_{e^{-r}}\left(\frac{\tau_2}{r} + t_1^2, \frac{\chi_2}{r} + h_1^2; \frac{\tau_1}{r} + t_2^2, \frac{\chi_1}{r} + h_2^2\right).$$

Proof of the lemma

Indeed :

$$\mathbb{E}_{e^{-r}}\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right]$$

is an alternate sum of $2|m|$ terms of two types :

Proof of the lemma

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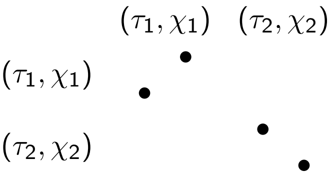

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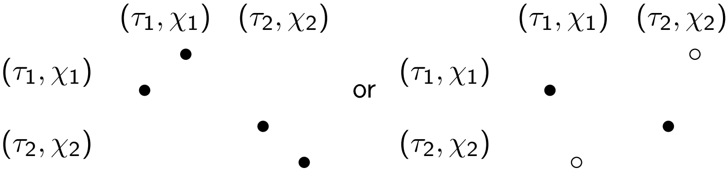

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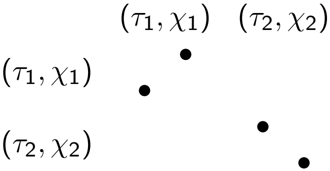
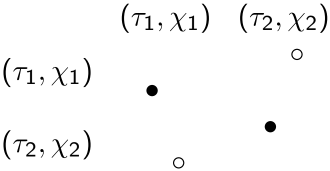


or



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$$\sum_{\sigma\in S(2|m|),\ \text{type } 1}$$

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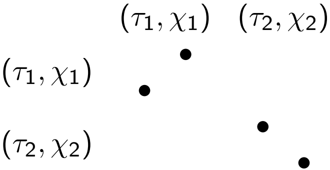

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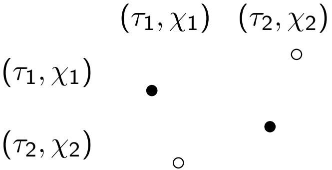

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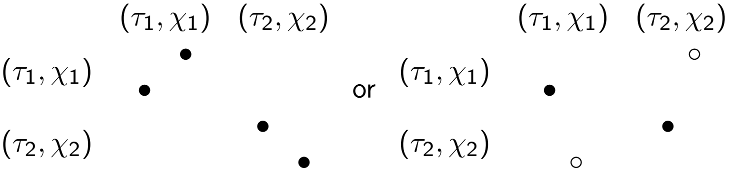

We have :

$$\sum_{\sigma\in S(2|m|),\ \text{type } 1} \bullet \ldots \bullet = \mathbb{E}_{e-r}\left[\mathbf{1}_{\frac{1}{r}(\tau_1,\chi_1)+m}\right]\mathbb{E}_{e-r}\left[\mathbf{1}_{\frac{1}{r}(\tau_2,\chi_2)+m}\right].$$

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Thus :

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Proof of the lemma

The factors "∘∘" of interests are quadruple integrals :

$$(1+O(1))\int_{z\in\gamma_{\tau_1}}\int_{w\in\gamma_{\tau_2}}\int_{z'\in\gamma_{\tau_2}}\int_{w'\in\gamma_{\tau_1}}\frac{dzdwdz'dw'}{(z-w)(z'-w')}$$

$$\exp\left(\frac{1}{r}\left(S(z;\tau_1,\chi_1)-S(w;\tau_2,\chi_2)+S(z';\tau_2,\chi_2)-S(w';\tau_1,\chi_1)\right)\right).$$

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We symmetrized the problem.

Deformation of the contours

We deform the contours γ_{τ_1} and γ_{τ_2} :

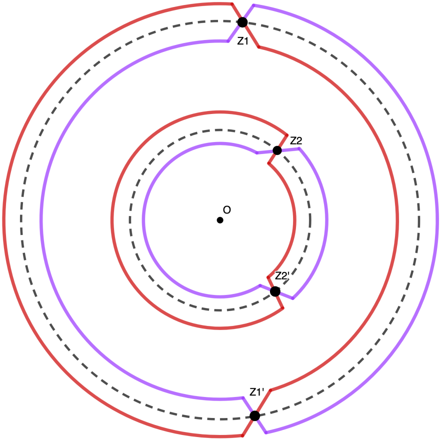

Figure – The contours $\gamma_{\tau_i}^{>}$ et $\gamma_{\tau_i}^{<}$

Proof of the lemma

On these contours, we have :

$$\Re S(z;\tau_1,\chi_1) - \Re S(w';\tau_1,\chi_1) < 0,\ z \in \gamma_{\tau_1}^{<},\ w' \in \gamma_{\tau_1}^{>},$$
$$\Re S(z';\tau_2,\chi_2) - \Re S(w;\tau_2,\chi_2) < 0,\ z' \in \gamma_{\tau_2}^{<},\ w \in \gamma_{\tau_2}^{>},$$

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$$\left| \frac{\exp\left(\frac{1}{r}\left(S(z;\tau_1,\chi_1) - S(w;\tau_2,\chi_2) + S(z';\tau_2,\chi_2) - S(w';\tau_1,\chi_1)\right)\right)}{(z-w)(z'-w')} \right|$$

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On these contours, we have :

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for all $\alpha < 1$.

Proof of the lemma

When $\tau_1 = \tau_2 = \tau$ the deformed contours look like :

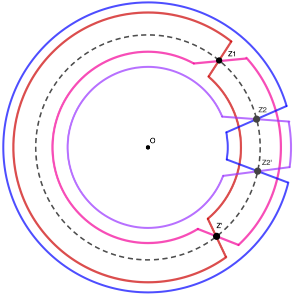

Figure – The contours $\gamma_{\tau}^{i,>}$ and $\gamma_{\tau}^{i,<}$

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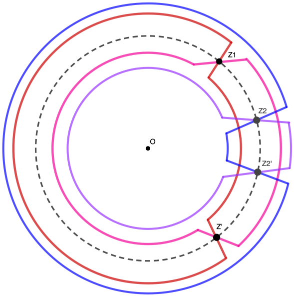

Figure – The contours $\gamma_\tau^{i,>}$ and $\gamma_\tau^{i,<}$

In this case, the new contours avoid the residues at $z = w$, $z' = w'$ and we have to integrate them to obtain the right asymptotics.

Proof of the lemma

This gives :

$$K_{e^{-r}}\left(\frac{1}{r}(\tau_1,\chi_1)+...;\frac{1}{r}(\tau_2,\chi_2)+...\right)$$
$$\times K_{e^{-r}}\left(\frac{1}{r}(\tau_2,\chi_2)+...;\frac{1}{r}(\tau_1,\chi_1)+...\right)$$
$$=\left(\int_{z\in\gamma_\tau^{1,<}}\int_{w\in\gamma_\tau^{2,>}}...+\int_w f(w;\tau,\chi_1,\chi_2)dw\right)$$
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$$f(w;\tau,\chi_1,\chi_2)\sim w^{\frac{1}{r}(\chi_2-\chi_1)+\cdots}.$$

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Integrating by parts, we obtain :

$$\left|\int_w f(w;\tau,\chi_1,\chi_2)dw\right| \leq C\frac{r}{|\chi_1-\chi_2|}\exp\left(\frac{\tau}{2r}(\chi_2-\chi_1)\right),$$

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By construction, the double integrals are equivalent to :

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and their product rapidly tends to 0. Developping the product, one obtains the result. □.

Thank you for your attention !