Scaling entropy and its applications

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Kolmogorov–Sinai entropy

Let (X, μ) be a measure space and $\xi = \{C_i\}_{i=1}^k$ be a measurable partition. The *entropy* $H(\xi)$ of ξ is

$$H(\xi) = -\sum_{i=1}^{k} \mu(C_i) \log \mu(C_i).$$

Now let T be a measure preserving transformation. We denote $\xi^n = \bigvee_{i=0}^{n-1} T^{-i}\xi$. The entropy of ξ with respect to T is the following value:

$$\boxed{h(T, \xi) = \lim_{n \to +\infty} \frac{1}{n} H(\xi^n).}$$

Kolmogorov–Sinai entropy of T is then:

$$h(T) = \sup\{h(\xi) \colon H(\xi) < +\infty\}.$$

Zero entropy systems

Kolmogorov–Sinai entropy solves the isomorphism problem for Bernoulli systems. However, it is still unclear how to separate systems with the same entropy.

Question

How to distinguish systems with $h(T) = 0$?

- For such transformations $H(\xi^n) = o(n), \ \forall \xi$.
- Naive way: change n to another $\phi(n)$ and consider

$$\lim_{n \to +\infty} \frac{1}{\phi(n)} H(\xi^n).$$

- In this case one can find a partition whose limit is infinite (Vershik, Ferenczi).

Vershik's approach

For the zero entropy case, A. Vershik proposed a new approach based on the dynamics of admissible semimetrics.

- Measurable partitions with finite entropy
  $\rightsquigarrow$ *measurable admissible semimetrics*
- Shannon entropy $H(\xi)$
  $\rightsquigarrow$ ε*–entropy of a semimetric* $\mathbb{H}_\varepsilon(\rho)$
- Refinement of partitions ξ^n
  $\rightsquigarrow$ *averaging of semimetrics* $T^n_{av}\rho$
- The existence of $\lim_{n\to+\infty}\frac{1}{n}H(\xi^n)$ for all ξ
  $\rightsquigarrow$ *boundedness of* $\frac{1}{\phi(n)}\mathbb{H}_\varepsilon(T^n_{av}\rho)$ *for all* ρ

Admissible semimetrics

Let $\rho \in L^1(X^2, \mu^2)$ be a non-negative symmetric measurable function which satisfies the triangle inequality.

Example

Let ξ be a measurable partition. The corresponding cut semimetric $\rho_\xi(x, y) = 0$ if x and y lie in the same cell of ξ and $\rho_\xi(x, y) = 1$ otherwise.

Define the ε*–entropy* $\mathbb{H}_\varepsilon(X, \mu, \rho)$ of ρ as follows.

Definition

Let k be the minimal integer such that there is a collection of k ε–balls with the total measure at least $1 - \varepsilon$. We set

$$\mathbb{H}_\varepsilon(X, \mu, \rho) = \log_2 k.$$

If there is no such k put $\mathbb{H}_\varepsilon(X, \mu, \rho) = +\infty$

Admissible semimetrics

The following conditions are equivalent (Vershik, Petrov, Zatitskiy '13)

1. The semimetric ρ is separable on a subset of full measure.
2. $\mathbb{H}_\varepsilon(X, \mu, \rho)$ is finite for any $\varepsilon > 0$.

Definition

In this case, the semimetric ρ is called *admissible*.

These properties are evident.

- If ρ is admissible then all its shifts $T^{-n}\rho(x, y) = \rho(T^n x, T^n y)$, are admissible as well.
- A finite averaging of an admissible semimetric is admissible

$$T^n_{av}\rho(x, y) = \frac{1}{n}\sum_{i=0}^{n-1} \rho(T^i x, T^i y), \qquad x, y \in X.$$

Scaling entropy sequence

Definition

A sequence $\{h_n\}$ is called *scaling* for a semimetric ρ if

$$h_n \asymp \mathbb{H}_\varepsilon(X, \mu, T^n_{av}\rho),$$

for all $\varepsilon > 0$ small enough.

Example

Let T be rotation of the unit circle $\mathbb{T}$, and ρ be the inner metric on $\mathbb{T}$. Then $h_n = 1$ is a scaling entropy sequence for ρ.

Semimetric ρ is called *generating* if there is $X_1 \subset X$ of full measure such that for any $x, y \in X_1$ there exists n with $\rho(T^n x, T^n y) > 0$.

- Any measurable metric is generating.
- If ξ is a generating partition then ρ_ξ is generating.

Invariance, subadditivity, and examples

Theorem (Zatitskiy '15)

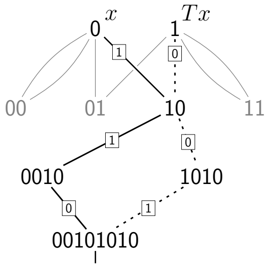
Let $\{h_n\}$ be a scaling entropy sequence for some admissible generating semimetric ρ. Then $\{h_n\}$ is scaling for any such semimetric.

Example (Vershik, Petrov, Zatitskiy '13)

- (X, μ, T) has pure point spectrum iff $h_n = 1$.
- Kolmogorov–Sinai entropy $h(T) > 0$ iff $h_n = n$.

Theorem (Petrov, Zatitskiy '15)

- *If scaling sequence exists, then there is an increasing subadditive function $f_n \asymp h_n$.*
- *Conversely, for any increasing subadditive f_n there is a system with $f_n \asymp h_n$.*



Unstable systems

Several natural questions arise here.

- Are there any *unstable* ergodic systems?
- If so, what is *unstable scaling entropy*?
- Subadditivity, examples?

Theorem (G.V. '20)

There is an ergodic system (X, μ, T) *and an admissible semimetric* ρ *such that* $\mathbb{H}_\varepsilon(X, \mu, T^n_{av}\rho)$ *essentially depends on* ε*. That is, for any* $\varepsilon > 0$ *there is* $\delta > 0$ *with*

$$\mathbb{H}_\varepsilon(X, \mu, T^n_{av}\rho) \lnsim \mathbb{H}_\delta(X, \mu, T^n_{av}\rho).$$

Scaling entropy

We are looking for an upgraded definition that coincides with the old one in the stable case and takes ε in consideration.

Definition

- We say that $\Phi(n,\varepsilon) \preceq \Psi(n,\varepsilon)$, if $\forall\varepsilon\ \exists\delta\ \Phi(n,\varepsilon) \lesssim \Psi(n,\delta)$.
- Two functions are equivalent if $\Phi \preceq \Psi$ and $\Psi \preceq \Phi$.
- We denote the equivalence class of Φ we denote by $[\Phi]$.

Given a semimetric ρ define $\Phi_\rho(n,\varepsilon) = \mathbb{H}_\varepsilon(X,\mu,T^n_{av}\rho)$.

Theorem (Zatitskiy '15)

Let ρ and ω be two admissible generating semimetrics. Then $[\Phi_\rho] = [\Phi_\omega]$.

Definition

Scaling entropy of a measure–preserving system is $\mathcal{H}(X,\mu,T) = [\Phi_\rho]$, where ρ is a generating admissible semimetric.

Subadditivity

The goal is to find a representative with some good properties.

1. Increasing in n: $\Phi(n, \varepsilon) \leqslant \Phi(m, \varepsilon)$ for $n < m$.
2. Subadditivity in n: $\Phi(n + m, \varepsilon) \leqslant \Phi(n, \varepsilon) + \Phi(m, \varepsilon)$.
3. Decreasing in ε: $\Phi(n, \varepsilon_1) \geqslant \Phi(n, \varepsilon_2)$ for $\varepsilon_1 < \varepsilon_2$.

Theorem (G.V. '20)

- *The class $\mathcal{H}(X, \mu, T)$ contains a subadditive monotone function.*
- *For any subadditive monotone function Φ there is an ergodic system (X, μ, T) such that $\mathcal{H}(X, \mu, T) = [\Phi]$.*

Thus the complete description of the possible values of the invariant is obtained.

Scaling entropy for group actions

The notion of scaling entropy can be generalized for group actions.

- Let G be *an amenable group* and $\lambda = \{F_n\}$ be a Følner sequence, i. e., for any $g \in G$
$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$
- For an admissible semimetric ρ define its averaging over F_n
$$G^n_{av}\rho(x,y) = \frac{1}{|F_n|}\sum_{g\in F_n} \rho(gx,gy), \qquad x,y \in X.$$
- Consider a function $\Phi_\rho(n,\varepsilon) = \mathbb{H}_\varepsilon(X,\mu,G^n_{av}\rho)$.
- And take the corresponding equivalence class $\mathcal{H}(X,\mu,T) = [\Phi_\rho]$.

The variational principle

There are two well–known characteristics of G–actions.

- The entropy $h(X, \mu, G)$ of a measure–theoretic action.
- The entropy $h_{top}(X, G)$ of a topological action.

They are related in the following way.

Theorem (The variational principle)

Let $G \curvearrowright X$ be a topological G–action. Then

$$h_{top}(X, G) = \sup_{\mu \in M_G(X)} h(X, \mu, G).$$

In particular, if the topological entropy is zero, then the measure theoretical entropy is zero as well.

Universal systems

Definition

A topological system (X, G) is called *universal* for some class $\mathcal{S}$ consisting of ergodic G–actions if

1. for any ergodic $\mu \in M_G(X)$ the system (X, μ, G) belongs to $\mathcal{S}$;
2. for any system $(Y, \nu, G) \in \mathcal{S}$ there exists $\mu \in M_G(X)$ such that (X, μ, G) and (Y, ν, G) are isomorphic.

Example

- The standard shift on $[0, 1]^G$ is universal for the class of all G–actions.
- (Krieger'70, Seward'18) Bernoulli shift on $\{1, \ldots, n\}^G$ is universal for all actions with $h < \log n$ (and one more*).
- (Downarowicz, Serafin'16) There exists a universal $\mathbb{Z}$–system for $h \in [0, \alpha)$ or $h \in [0, \alpha]$, $\alpha > 0$.

Universal zero entropy system

Question (B. Weiss)

Does there exist a system (X, G) which is universal for the class of all ergodic measure–preserving actions of zero entropy?

For the case of $\mathbb{Z}$, the negative answer was given by J. Serafin ('13). However, this question is still open for general amenable groups. Our main result is the following theorem.

Theorem (G.V. '20)

Let $G \curvearrowright X$ be a continuous action of a non–periodic amenable group. Assume that for any ergodic zero entropy system (Y, ν, G) there exists $\mu \in M_G(X)$ such that

$$(X, \mu, G) \cong (Y, \nu, G).$$

Then the topological entropy of (X, G) is positive.

Actions of almost complete growth

Definition

We say that (G, λ) *admits actions of almost complete growth* if for any non–negative function $\phi(n) = o(|F_n|)$ there exists a measure–preserving system (X, μ, G) such that for any $\Phi \in \mathcal{H}(X, \mu, G, \lambda)$

- $\Phi(n, \varepsilon) = o(|F_n|)$,
- $\Phi(n, \varepsilon) \not\lesssim \phi(n)$.

The first condition is equivalent to the fact that the measure entropy is zero.

1. We prove that the main theorem holds for any such group.
2. The main part: every non–periodic amenable group admits ergodic actions of almost complete growth with respect to arbitrary Følner equipment.
   - Construct almost complete actions for the group $\mathbb{Z}$.
   - Apply coinduction from a subgroup to the whole G.

Thank you for your attention!