Limit theorems for Lévy flights on a 1D Lévy random medium

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2. The Model
   - Description of the model and goal
   - The space $\mathcal{D}$ of càdlàg functions
   - Metrics on $\mathcal{D}$: J_1 and J_2
3. Results
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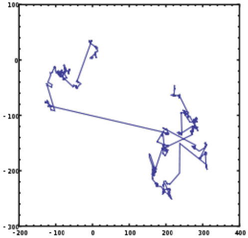
Lévy Flights

What is a Lévy Flight?

Idea: *A set of random short movements connected by infrequent longer ones.*

There are many situations which can be modeled in terms of **Lévy flights**:

- Human mobility
- Animal foraging
- Internet browsing



More formally: *discrete-time random walk with heavy-tailed instantaneous jumps (infinite variance).*

Lévy Flights on random medium

What is a Lévy random medium?

A stochastic point process, in some space, where the distances between nearby points have heavy-tailed distributions

- Consider a random medium in the real line: a sequence of random points $\omega = (\omega_k, k \in \mathbb{Z})$ whose nearest-neighbor distances are i.i.d. and heavy-tailed random variables;
- Consider a random walk $Y = (Y_n, n \in \mathbb{N})$ which takes place on ω.
- Y ⇨ **Lévy flight on a 1D Lévy random medium**

Stable distributions

Definition (α-stable distribution)

The distribution of a non-degenerate random variable $W^{(\alpha)}$ is $\alpha-$ stable if there exist constants $a_n > 0$ and b_n such that for any $n \geq 2$, if $W_1^{(\alpha)}, W_2^{(\alpha)}, \cdots$ are independent random variables distributed like $W^{(\alpha)}$ and $S_n := \sum_{i=1}^{n} W_i^{(\alpha)}$, then

$$\frac{S_n - b_n}{a_n} \stackrel{d}{=} W^{(\alpha)}, \qquad \alpha \in (0, 2]. \tag{1}$$

- $a_n = n^{1/\alpha}$ for some $\alpha \in (0, 2]$;
- if $\alpha \in (1, 2]$, $W^{(\alpha)}$ has **finite mean**
- The case $\alpha = 2$ corresponds to the **Gaussian distribution**
- Stable laws have power tails such as

$$\mathbb{P}(W^{(\alpha)} > x) \sim C_\alpha x^{-\alpha} \qquad x \to +\infty, \quad \alpha \in (0, 2). \tag{2}$$

Definition (Normal domain of attraction of an α-stable distribution)

A random variable ξ belongs to the *normal domain of attraction of an* $\alpha-$ *stable distribution* if there exist constants $c_n > 0$ and b_n such that

$$\frac{S_n - b_n}{c_n} \xrightarrow[n\to+\infty]{d} W^{(\alpha)}, \tag{3}$$

where $S_n := \sum_{i=1}^{n} \xi_i$ is a sum of i.i.d copies of ξ and $W^{(\alpha)}$ is an $\alpha-$ stable random variable.

- If $\alpha \in (0,1)$ then [1]

$$\frac{S_n}{n^{1/\alpha}} \xrightarrow{d} W^{(\alpha)}. \tag{4}$$

- If $\alpha \in (1,2)$ then

$$\frac{S_n - n\mathbb{E}(\xi)}{n^{1/\alpha}} \xrightarrow{d} W^{(\alpha)} \tag{5}$$

$$\frac{S_n}{n} \xrightarrow{a.s} \mathbb{E}(\xi). \tag{6}$$

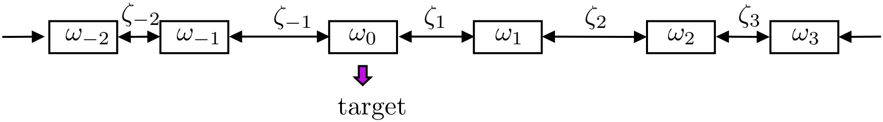
[1] Feller [3, Theorem XVII.5.3]

The Model: Lévy-Lorentz gas (Barkai, Fleurov, Klafter ['00])

Random medium ω

- Let $\zeta = (\zeta_j, j \in \mathbb{Z})$ be a sequence of i.i.d. positive random variables;
- assume that ζ belongs to the normal domain of attraction of a β-stable distribution, with $0 < \beta < 2$;
- Define the point process associated to $(\zeta_j, j \in \mathbb{Z})$ as:

$$\omega_0 := 0, \qquad \omega_k := \begin{cases} \sum_{i=1}^{k} \zeta_i & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -\sum_{i=k}^{-1} \zeta_i & \text{if } k < 0. \end{cases} \tag{7}$$

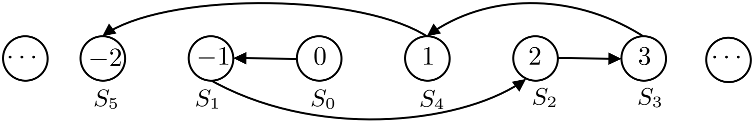


The Model

Underlying random walk S

- Let $\xi = (\xi_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables;
- assume that ξ_i belongs to the normal domain of attraction of a α-stable distribution, with $0 < \alpha < 2$;
- define the underlying random walk associated to $(\xi_i, i \in \mathbb{N})$ as:

$$S_0 := 0, \qquad S_n := \sum_{i=1}^{n} \xi_i \quad \text{for } n \in \mathbb{N}^+. \qquad (8)$$



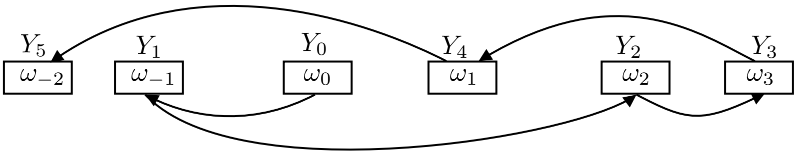
$$\xi_i = S_i - S_{i-1}$$

The Model

Random walk on the point process: $Y = (Y_n, n \in \mathbb{N})$, where

$$Y_n := \omega_{S_n} \equiv \omega \circ S(n), \quad n \in \mathbb{N}. \tag{9}$$

- Y_n performs the same jumps as S_n, but on the points of ω.
- if $S = (0, -1, 2, 3, 1, -2, ...)$ is a realization of the underlying random walk, the corresponding random walk on the point process will be $Y = (\omega_0, \omega_{-1}, \omega_2, \omega_3, \omega_1, \omega_{-2}, \dots)$.



Processes that arise from ω_n and S_n

According to the domain of α and β we define the following rescaled processes: for $s, t \geq 0$

- $\boldsymbol{\alpha \in (0,1)}$ **:**
  $$\hat{S}^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{n^{1/\alpha}}$$
- $\boldsymbol{\alpha \in (1,2)}$ **:**
  $$\bar{S}^{(n)}(t) := \frac{S_{\lfloor nt \rfloor}}{n}$$
- $\boldsymbol{\beta \in (0,1)}$ **:**
  $$\hat{\omega}^{(n)}(s) := \frac{\omega_{\lfloor ns \rfloor}}{n^{1/\beta}}$$
- $\boldsymbol{\beta \in (1,2)}$ **:**
  $$\bar{\omega}^{(n)}(s) := \frac{\omega_{\lfloor ns \rfloor}}{n}$$

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- $\boldsymbol{\beta \in (0,1)}$ **:** $\hat{\omega}^{(n)}(s) := \frac{\omega_{\lfloor ns \rfloor}}{n^{1/\beta}}$
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GOAL

convergence in distribution of

1) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha}} = \bar{\omega}^{(n)} \circ \hat{S}^{(n)}$, $\boldsymbol{\alpha \in (0,1), \beta \in (1,2)}$

2) $\frac{Y_{\lfloor nt \rfloor}}{n} = \bar{\omega}^{(n)} \circ \bar{S}^{(n)}$, $\boldsymbol{\alpha \in (1,2), \beta \in (1,2)}$

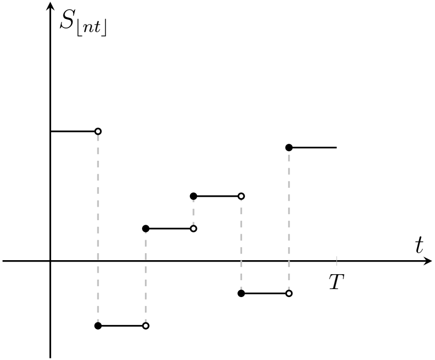
3) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\beta}} = \hat{\omega}^{(n)} \circ \bar{S}^{(n)}$, $\boldsymbol{\alpha \in (1,2), \beta \in (0,1)}$

4) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha\beta}} = \hat{\omega}^{(n)} \circ \hat{S}^{(n)}$, $\boldsymbol{\alpha \in (0,1), \beta \in (0,1)}$

$$\left(\frac{Y_{\lfloor nt \rfloor}}{n^{\gamma}}, t \in [0, +\infty) \right) \xrightarrow{d} ?$$

Space $\mathcal{D}$ of càdlàg functions

We denote by $\mathcal{D}([a,b]) = \mathcal{D}([a,b],\mathbb{R})$ the space of functions $f : [a,b] \mapsto \mathbb{R}$ which are continuous from the right at every point and such that $\lim_{s \to t^-} f(s)$ exists for all $t \in [a,b]$ (**càdlàg**=**c**ontinu **à** **d**roite, **l**imites **à** **g**auche)



Notation:

- $\mathcal{D} := \mathcal{D}(\mathbb{R},\mathbb{R})$;
- $\mathcal{D}^+ := \mathcal{D}(\mathbb{R}^+,\mathbb{R})$;

The space $\mathcal{D}$ includes all continuous functions

Metrics on $\mathcal{D}$: the idea of J_1

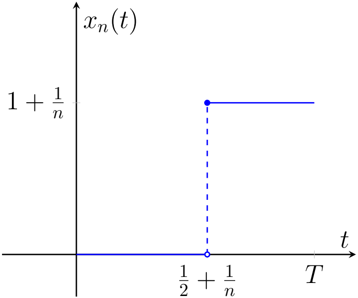
Functions are close if they are uniformly close after allowing continuous small perturbations of time (the function argument).

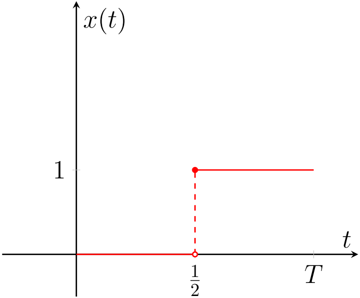
Example: Consider

- $x_n(t) = (1+n^{-1})\mathbb{1}_{[2^{-1}+n^{-1},T]}(t)$
- $x(t) = \mathbb{1}_{[2^{-1},T]}(t)$

$$x_n \to x \quad \text{in} \quad (\mathcal{D}^+, J_1) \tag{10}$$

$$||x_n - x||_\infty \nrightarrow 0 \tag{11}$$





Metrics on $\mathcal{D}$: the idea of J_2

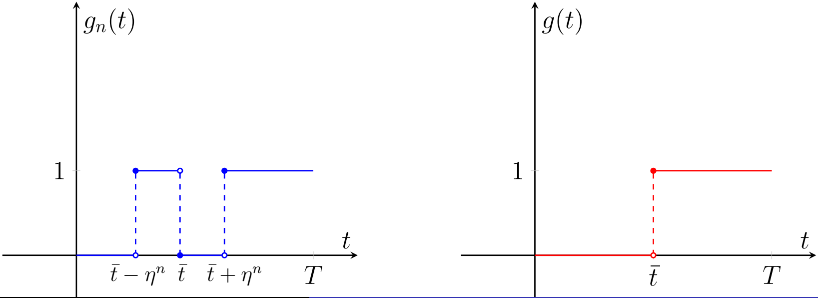
Functions are close if they are uniformly close after allowing small perturbations of time, not necessarily continuous.

Example: For $\bar{t} \in [0, T]$ and $\eta \in (0, 1)$ consider

- $g_n(t) = \mathbb{1}_{[\bar{t}-\eta^n,\bar{t})}(t) + \mathbb{1}_{[\bar{t}+\eta^n,T]}(t)$
- $g(t) = \mathbb{1}_{[\bar{t},T]}(t)$

$$g_n \to g \quad \text{in} \quad (\mathcal{D}^+, J_2) \tag{12}$$

$$g_n \nrightarrow g \quad \text{in} \quad (\mathcal{D}^+, J_1) \tag{13}$$



Formal definition of J_1 and J_2

Definition

Let $\{f_n\}_{n\in\mathbb{N}}, f$ be càdlàg functions defined on a closed interval $[0, T]$. The sequence f_n is said to converge to f in the J_1 (respectively J_2) Skorokhod topology if there exists a sequence of homeomorphisms (respectively bijections) $\lambda_n : [0, T] \mapsto [0, T]$ such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |f_n \circ \lambda_n(t) - f(t)| = 0, \tag{14}$$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = 0. \tag{15}$$

$\boldsymbol{U} > \boldsymbol{J_1} > \boldsymbol{J_2}$

Processes that arise from ω_n and S_n

For $t, s \geq 0$

- $\boldsymbol{\alpha \in (0,1)}$ **:** $\hat{S}^{(n)}(t) := \dfrac{S_{\lfloor nt \rfloor}}{n^{1/\alpha}}$
- $\boldsymbol{\alpha \in (1,2)}$ **:** $\bar{S}^{(n)}(t) := \dfrac{S_{\lfloor nt \rfloor}}{n}$
- $\boldsymbol{\beta \in (0,1)}$ **:** $\hat{\omega}^{(n)}(s) := \dfrac{\omega_{\lfloor ns \rfloor}}{n^{1/\beta}}$
- $\boldsymbol{\beta \in (1,2)}$ **:** $\bar{\omega}^{(n)}(s) := \dfrac{\omega_{\lfloor ns \rfloor}}{n}$

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3) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\beta}} = \hat{\omega}^{(n)} \circ \bar{S}^{(n)}$, $\boldsymbol{\alpha \in (1,2), \beta \in (0,1)}$

4) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha\beta}} = \hat{\omega}^{(n)} \circ \hat{S}^{(n)}$, $\boldsymbol{\alpha \in (0,1), \beta \in (0,1)}$

$$\left(\frac{Y_{\lfloor nt \rfloor}}{n^{\gamma}}, \; t \in [0, +\infty) \right) \xrightarrow{d} ?$$

Results

Define

- $\mu := \mathbb{E}(\xi)$ when $\alpha \in (1,2)$
- $\nu := \mathbb{E}(\zeta)$ when $\beta \in (1,2)$

Theorem (A.Bianchi, G.Bet, M.Lenci, E.M, S.Stivanello)

Let $\beta \in (1,2)$. Assume $\alpha \in (0,1)$ or $\alpha \in (1,2)$ and $\mu = 0$. Define, for every $t \geq 0$

$$\hat{Y}^{(n)}(t) := \bar{\omega}^{(n)} \circ \hat{S}^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha}}. \tag{16}$$

Then

$$\hat{Y}^{(n)} \xrightarrow{d} \nu W^{(\alpha)} \qquad in\ (\mathcal{D}^+, J_1), \tag{17}$$

where $W^{(\alpha)}$ is an α-stable process with $W^{(\alpha)}(0) = 0$.

- $\hat{S}^{(n)}(\cdot) := \frac{S_{\lfloor n \cdot \rfloor}}{n^{1/\alpha}}$
- $\bar{\omega}^{(n)}(\cdot) := \frac{\omega_{\lfloor n \cdot \rfloor}}{n}$
- $\hat{S}^{(n)} \xrightarrow{d} W^{(\alpha)}$ in $(\mathcal{D}^+, J_1)$
- $\bar{\omega}^{(n)} \xrightarrow{a.s} \nu \mathrm{id}$ in $(\mathcal{D}, J_1)$

Theorem (A.Bianchi, G.Bet, M.Lenci, E.M, S.Stivanello)

Assume $\beta \in (1,2)$, $\alpha \in (1,2)$ *and* $\mu \neq 0$, *and define, for every* $t \geq 0$,

$$\bar{Y}^{(n)}(t) := \bar{\omega}^{(n)} \circ \bar{S}^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{n}. \tag{18}$$

Then

$$\bar{Y}^{(n)} \xrightarrow{d} \nu\mu\,\mathrm{id} \qquad \text{in } (\mathcal{D}^{+}, J_1). \tag{19}$$

- $\bar{S}^{(n)}(\cdot) := \frac{S_{\lfloor n \cdot \rfloor}}{n}$
- $\bar{\omega}^{(n)}(\cdot) := \frac{\omega_{\lfloor n \cdot \rfloor}}{n}$

- $\bar{S}^{(n)} \overset{a.s}{\to} \mu\mathrm{id}$ in $(\mathcal{D}^{+}, J_1)$
- $\bar{\omega}^{(n)} \overset{a.s}{\to} \nu\mathrm{id}$ in $(\mathcal{D}, J_1)$

$\alpha \in (1,2), \beta \in (0,1)$:

- $\bar{S}^{(n)} \stackrel{a.s}{\to} \mu \mathrm{id} \quad$ in $(\mathcal{D}^+, J_1)$
- $\hat{\omega}^{(n)} \stackrel{d}{\to} Z^{(\beta)} \quad$ in $(\mathcal{D}, J_1)$

$\alpha \in (1,2), \beta \in (0,1)$:

- $\bar{S}^{(n)} \stackrel{a.s}{\rightarrow} \mu \mathrm{id}$ in $(\mathcal{D}^+, J_1)$
- $\hat{\omega}^{(n)} \stackrel{d}{\rightarrow} Z^{(\beta)}$ in $(\mathcal{D}, J_1)$

Theorem (A.Bianchi, G.Bet, M.Lenci, E.M, S.Stivanello)

Let $\beta \in (0,1)$ *and* $\alpha \in (1,2)$ *with* $\mu > 0$. *Define, for every* $t \geq 0$

$$\hat{Y}^{(n)}(t) := \hat{\omega}^{(n)} \circ \bar{S}^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{n^{1/\beta}} \tag{20}$$

Then, the following convergence holds with respect to the J_2 *Skorokhod topology*

$$\hat{Y}^{(n)} \xrightarrow{d} \mu^{1/\beta} Z^{(\beta)} \qquad \text{in } (\mathcal{D}^+, J_2). \tag{21}$$

Theorem (A.Bianchi, G.Bet, M.Lenci, E.M, S.Stivanello)

Let $\beta \in (0,1)$ and $\alpha \in (0,1)$ or $\alpha \in (1,2)$ and $\mu = 0$. Define, for every $t \geq 0$,

$$\hat{Y}^{(n)}(t) := \hat{\omega}^{(n)} \circ \hat{S}^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha\beta}}, \tag{22}$$

and let $W^{(\alpha)}$ be a Lévy α-stable process with $W^{(\alpha)}(0) = 0$. Then the finite dimensional distributions of $\hat{Y}^{(n)}$ converge to those of $Z^{(\beta)} \circ W^{(\alpha)}$, that is, for any $m \in \mathbb{N}^+$, $t_1, \ldots, t_m \in \mathbb{R}^+$

$$(\hat{Y}^{(n)}(t_1), \cdots, \hat{Y}^{(n)}(t_m)) \xrightarrow{d} (Z^{(\beta)}(W^{(\alpha)}(t_1)), \cdots, Z^{(\beta)}(W^{(\alpha)}(t_m))), \tag{23}$$

- $\hat{S}^{(n)}(\cdot) := \frac{S_{\lfloor n\cdot \rfloor}}{n^{1/\alpha}}$
- $\hat{\omega}^{(n)}(\cdot) := \frac{\omega_{\lfloor n\cdot \rfloor}}{n^{1/\beta}}$

- $\hat{S}^{(n)} \xrightarrow{d} W^{(\alpha)}$ in $(\mathcal{D}^+, J_1)$
- $\hat{\omega}^{(n)} \xrightarrow{d} Z^{(\beta)}$ in $(\mathcal{D}, J_1)$

Theorem (A.Bianchi, G.Bet, M.Lenci, E.M, S.Stivanello)

Let $\alpha, \beta \in (1,2)$ *with* $\mu > 0$ *and* $\bar{Y}^{(n)}(t) := \bar{\omega}^{(n)} \circ \bar{S}^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{n}$.

1. *When* $\alpha < \beta$

$$\frac{n(\bar{Y}^{(n)} - \nu\mu\mathrm{id})}{n^{1/\alpha}} \xrightarrow{d} \nu W^{(\alpha)} \qquad \text{in } (\mathcal{D}^+, J_1), \tag{24}$$

2. *When* $\alpha > \beta$ *we have*

$$\frac{n(\bar{Y}^{(n)} - \nu\mu\mathrm{id})}{n^{1/\beta}} \xrightarrow{d} \mu^{1/\beta} Z^{(\beta)} \qquad \text{in } (\mathcal{D}^+, J_2), \tag{25}$$

3. *When* $\alpha = \beta$ *we have*

$$\frac{n(\bar{Y}^{(n)} - \nu\mu\mathrm{id})}{n^{1/\alpha}} \xrightarrow{d} \mu^{1/\alpha} Z^{(\alpha)} + \nu W^{(\alpha)} \qquad \text{in } (\mathcal{D}^+, J_2), \tag{26}$$

with $Z^{(\alpha)}$, $W^{(\alpha)}$ *independent* $\alpha-$*stable processes.*

Conclusions

- $\boldsymbol{\alpha \in (0,1)}$ **:**
  $\hat{S}^{(n)}(\cdot) := \dfrac{S_{\lfloor n \cdot \rfloor}}{n^{1/\alpha}}$
- $\boldsymbol{\alpha \in (1,2)}$ **:**
  $\bar{S}^{(n)}(\cdot) := \dfrac{S_{\lfloor n \cdot \rfloor}}{n}.$
- $\boldsymbol{\beta \in (0,1)}$ **:**
  $\hat{\omega}^{(n)}(\cdot) := \dfrac{\omega_{\lfloor n \cdot \rfloor}}{n^{1/\beta}}.$
- $\boldsymbol{\beta \in (1,2)}$ **:**
  $\bar{\omega}^{(n)}(\cdot) := \dfrac{\omega_{\lfloor n \cdot \rfloor}}{n}.$

Results:

1) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha}} = \bar{\omega}^{(n)} \circ \hat{S}^{(n)} \xrightarrow{\text{d}} \nu W^{(\alpha)}$ in $(\mathcal{D}^+, J_1)$

2) $\frac{Y_{\lfloor nt \rfloor}}{n} = \bar{\omega}^{(n)} \circ \bar{S}^{(n)} \xrightarrow{\text{d}} \nu \mu \mathrm{id}$ in $(\mathcal{D}^+, J_1)$

3) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\beta}} = \hat{\omega}^{(n)} \circ \bar{S}^{(n)} \xrightarrow{\text{d}} \mu^{1/\beta} Z^{(\beta)}$ in $(\mathcal{D}^+, J_2)$

4) $\frac{Y_{\lfloor nt \rfloor}}{n^{1/\alpha\beta}} = (\hat{\omega}^{(n)} \circ \hat{S}^{(n)}(t_1), \cdots, \hat{\omega}^{(n)} \circ \hat{S}^{(n)}(t_m)) \xrightarrow{\text{d}}$
$(Z^{(\beta)}(W^{(\alpha)}(t_1)), \cdots, Z^{(\beta)}(W^{(\alpha)}(t_m))),$

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